Deformations of flat conformal structures for hyperbolic 3-manifolds

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Definition: Suppose M is a smooth, orientable n-manifold. A *flat conformal structure* on M is a maximal atlas of charts $\phi_\alpha : U_\alpha \to \mathbf{S}^n$ such that the transition maps are restrictions of Möbius transformations.

From such a structure we get a developing map $dev : \tilde{M} \to \mathbf{S}^n$ and a holonomy representation $\rho_0 : \pi_1(M) \to M\ddot{o}b^+(\mathbf{S}^n)$.

$$\begin{aligned} SO_0(n+1,1) &\cong Isom^+\mathbf{H}^{n+1} \\ &\cong Isom^+_\uparrow \mathbf{S}^{n+1}_1 \\ &\cong M\ddot{o}b^+(\mathbf{S}^n). \end{aligned}$$

Examples

1. In the case $n = 2$ we have $\mathbf{S}^2 = \mathbf{CP}^1$ and $SO_0(3,1) \cong PSL(2,\mathbf{C})$, so a flat conformal structure coincides with the classical notion of a projective structure on a Riemann surface.

2. When $n \geq 3$, a manifold with a conformally flat Riemannian metric has a compatible flat conformal structure (Liouville).

3. In particular, all constant curvature manifolds have canonically associated flat conformal structures, e.g. view $\mathbf{H}^n$ as a hemisphere of $\mathbf{S}^n$ which is invariant under the subgroup

$$SO_0(n,1) \subset SO_0(n+1,1).$$

4. Other 3-dimensional geometries:

Yes: $\mathbf{H}^2 \times \mathbf{R}$, $\mathbf{S}^2 \times \mathbf{R}$
No: Nil, $Solv$ (Goldman)
Maybe: $UT(\mathbf{H}^2)$ (Gromov, Lawson, Thurston)

Deformation Spaces

Deformations of complete hyperbolic structures of finite volume are well-understood, in the case $n = 2$ by Teichmüller theory and for $n \geq 3$ by Mostow Rigidity.

Main Problem: Describe the deformation space of flat conformal structures near the canonical structure coming from a finite volume hyperbolic manifold.

Let $\pi_1(M) \cong \Gamma \subset SO_0(n, 1)$ be the inclusion of a lattice. Then, at least locally, the deformation space of flat conformal structures is parameterized by the representation variety

$$R(\pi_1(M), SO_0(n + 1, 1))$$

(representations up to conjugation) near Γ.

Once again the case $n = 2$ is well-understood; it reduces to the theory of quasi-Fuchsian deformations of Fuchsian groups.

Motivation

1. The existence of deformations is often reflected the geometry and topology of the hyperbolic manifold, e.g. most known examples of deformations arise from the existence of a closed, embedded, totally geodesic hypersurface (“bending”).

2. Conjecture [Menasco-Reid] No hyperbolic knot complement contains a closed, embedded, totally geodesic hypersurface.

3. An arbitrary flat conformal structure on M yields, in a canonical way, a hyperbolic metric and a dual de Sitter metric on $M \times \mathbf{R}$.

Theorem [S, 1996] Every de Sitter spacetime arises in this way; i.e. from a flat conformal structure at timelike infinity.

Sketch: Given a spacelike immersion D of $\tilde{M}$ into de Sitter space, there is a maximal domain of dependence extending D with a causal horizon $H^+(M)$. The general theory (Hawking/Ellis) says that this horizon is made up of of future or past complete null rays, from which we deduce the local convexity of $H^+(M)$. The ϵ-timelike distant hypersurfaces are then strictly convex, which gives an immersion into the sphere at timelike infinity via the Gauss map.

This theorem shows that the deformation space of de Sitter metrics on $M \times \mathbf{R}$ is determined by the deformation space of flat conformal structures on M. This was our original motivation for studying the latter.

Remark: We also obtain "no topology-change" results for de Sitter spacetimes using this classification.

Cohomology

Returning to the Main Problem, we consider the case $n = 3$, and write $\pi = \pi_1(M)$ for a closed hyperbolic 3-manifold M.

The Zariski tangent space to $R(\pi, SO_0(4,1))$ at a representation ρ_0 is given by the group cohomology:

$$\begin{aligned} H^1(\pi, so(4,1)) &\cong H^1(\pi, so(3,1)) \oplus H^1(\pi, \mathbf{R}^4_1) \\ &\cong H^1(\pi, \mathbf{R}^4_1). \end{aligned}$$

Here the coefficients lie in the Lie algebra $so(4,1)$ which is made into a $\mathbf{Z}\pi$-module via ρ_0 and the adjoint representation. We call a non-zero cocycle in $H^1(\pi, \mathbf{R}^4_1)$ an *infinitesimal deformation* of the flat conformal structure on M.

Fibonacci manifolds and Turk's head links

The Fibonacci groups are defined by:

$$F(2,n) = \{a_1, \ldots, a_n \mid a_i a_{i+1} = a_{i+2}\}.$$

The groups $F(2,2m)$ for $m \geq 4$ are the fundamental groups of closed hyperbolic 3-manifolds F_m (Helling, Kim, Mennicke, 1989).

These manifolds can be described as branched covers of links in $\mathbf{S}^3$ in several different ways, e.g. F_m is the m-fold cyclic branched cover of the figure-eight knot, and also the 2-fold branched cover of the Turk's head link B_m:

Main Result

Theorem [S, 1997] For all $m \geq 4$,

$$\dim_{\mathbf{R}} H^1(F_m, \mathbf{R}^4_1) = 2.$$

Idea of proof: Use the description of F_m as the 2-fold branched covering of a closed 3-braid to get an upper bound on the dimension of this cohomology group. The symmetry of the presentation gives an automorphism of $F(2, 2m)$ which induces a certain linear transformation of $\mathbf{R}^4_1 \times \mathbf{R}^4_1$. Eigenvalues which are $2m$-roots of unity determine cocycles, and these can be calculated directly.

Remarks

1. Kapovich has conjectured that for a closed, hyperbolic 3-orbifold M, this cohomology group will be non-trivial if and only if M contains a closed, embedded, quasi-Fuchsian suborbifold. But the manifold F_4 is non-Haken, and contains no closed immersed totally geodesic surface.

2. This also can be contrasted with the theorem of Ghys and Rajan that the complex structure on $\Gamma\backslash SL(2, \mathbf{C})$ deforms if and only if the first Betti number $\beta_1(\Gamma) \neq 0$.

3. The volume of F_4 is the same as the figure-eight knot $(2.02988\ldots)$. We conjecture that it is the smallest hyperbolic 3-manifold with infinitesimal deformations (previous best was $3.226\ldots$).

4. Similar non-vanishing theorems are true for the Turk's head links (in parabolic cohomology); nevertheless, the Menasco-Reid conjecture can be verified directly for these examples. True up to ten crossing knots.