3-manifolds which are spacelike slices of flat spacetimes

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Statement of Main Result

We will be considering spacetimes homeomorphic to $M^3 \times \mathbf{R}$ where M^3 is a compact, connected, topological 3-manifold without boundary. The slices $M^3 \times \{t\}$ will always be spacelike.

A fundamental question is to determine the possible topologies of the universe after imposing various (usually strong) conditions on the spacetime metric. Only in the last few years has 3-manifold topology advanced enough provide answers to this question in certain cases:

Main Theorem. M is a spacelike slice of a flat spacetime if and only if M is hyperbolic (admits a Riemannian metric of constant curvature -1) or M is finitely covered by $\Sigma^2 \times \mathbf{S}^1$ where Σ is a closed orientable surface other than $\mathbf{S}^2$.

The condition that M is hyperbolic is a purely topological condition – all that is meant is that M *admits* a hyperbolic metric, not that the induced Riemannian metric on some slice is hyperbolic (indeed this is usually not the case).

Related results

Theorem (UCLA thesis, 1996). M^n is a spacelike slice of a de Sitter spacetime if and only if M admits a conformally flat Riemannian metric.

Remarks

1. This (de Sitter) result works in all dimensions.

2. Admitting a conformally flat Riemannian metric is a non-trivial topological constraint when $n \geq 3$.

3. More is true: the moduli space of de Sitter domains of dependence $M \times \mathbf{R}$ is parameterized by the moduli space of conformally flat Riemannian metrics on M.

4. In contrast, the main theorem is special to $3+1$ and makes no statement about the moduli space of flat metrics on $M \times \mathbf{R}$. In the course of the proof, though, we will classify all holonomy representations $\pi_1(M) \to ISO(3,1)$.

5. The moduli space in the flat $2+1$ dimensional case was worked out by Geoff Mess in 1990.

Some perspective

Thurston discovered in the 1970's that "most" 3-manifolds are hyperbolic. The expected characterization is that M is hyperbolic if and only if M is irreducible (i.e. every smoothly embedded sphere bounds a ball), $\pi_1(M)$ is infinite, and $\pi_1(M)$ contains no $\mathbf{Z} \oplus \mathbf{Z}$ subgroup.

Nevertheless, there are lots of other possibilities for 3-manifolds which we must prove do not occur as slices:

1. Manifolds not covered by $\mathbf{R}^3$ (like $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{S}^1, \ldots$)

2. Some *Seifert fiber spaces*. These are 3-manifolds which are foliated by circles. Of these, some are finitely covered by $\Sigma \times \mathbf{S}^1$ while others are not (e.g. the unit tangent bundle of a hyperbolic surface). A key element in the proof is to exclude these non-trivial Seifert fiber spaces.

3. Solv-manifolds, graph manifolds, etc.

While it is instructive to think about the main theorem in terms of excluding various families, one cannot prove it this way since 3-manifolds are not classified.

Realizing 3-manifolds as slices

This is the easier half of the theorem. The 3-manifolds which arise in the statement all admit nice Riemannian metrics with (metric) universal cover either $\mathbf{H}^3$, $\mathbf{E}^3$, or $\mathbf{H}^2 \times \mathbf{R}$.

It suffices to realize these "geometries" inside $\mathbf{R}^4_1$:

Proof of the converse

A flat metric on $M \times \mathbf{R}$ yields a spacelike immersion $d : \tilde{M} \to \mathbf{R}^4_1$ and a holonomy representation $\phi : \pi_1(M) \to ISO(3,1)$ with discrete image.

A result of S. Harris shows that d is an achronal embedding, in particular that $\tilde{M}$ is homeomorphic to $\mathbf{R}^3$.

The rest of the proof amounts to classifying possibilities for the discrete groups $\phi(\pi_1(M))$ in $ISO(3,1)$ and then using powerful "homotopy equivalence implies homeomorphism" results from 3-manifold topology. In other words, the 3-manifolds which arise as slices are all determined by their fundamental groups.

Holonomy representations

We have

$$1 \to \mathbf{R}^4_1 \to ISO(3,1) \to SO(3,1) \to 1$$

and so the holonomy $\Gamma = \phi(\pi_1(M))$ has a *translational part* which is a discrete subgroup of $\mathbf{R}^4_1$, isomorphic to $\mathbf{Z}^k$ for $k = 0, 1, 2, 3$. We have:

$$1 \to \mathbf{Z}^k \to \Gamma \to L(\Gamma) \to 1$$

Statements below are "up to finite covers":

Case $k = 0$: Either $\Gamma = \mathbf{Z}^3$ or Γ embeds as a discrete subgroup of $SO(3,1)$. In the first case, M is a 3-torus (Waldhausen), and in the second case M is hyperbolic (Gabai, Meyerhoff, and N. Thurston).

Case $k = 1$: The Seifert Fiber Space Conjecture, recently resolved by Gabai, Casson-Jungreis, and Mess, implies in this case that M is a Seifert fiber space. The key step is to show it is covered by $\Sigma \times \mathbf{S}^1$.

Case $k = 2$: A theorem of Stallings implies that M fibers over the circle with torus fibers. A straightforward argument shows that in fact M is a 3-torus.

Case $k = 3$: M is a 3-torus (Waldhausen).