Flat conformal structures and constant curvature spacetimes

Kevin P. Scannell
Saint Louis University

Some Definitions

Minkowski space $\mathbf{R}_1^n$ is just $\mathbf{R}^n$ equipped with the standard signature $(n-1,1)$ inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + \ldots + v_{n-1} w_{n-1} - v_n w_n.$$

A *Lorentzian manifold* is defined like a Riemannian manifold, except the tangent spaces resemble Minkowski space instead of Euclidean space. Lorentzian manifolds are models for general relativity:

$\mathbf{v}$ is *spacelike* if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$

$\mathbf{v}$ is *timelike* if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$

$\mathbf{v}$ is *lightlike* if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.

A *world line* is a path in a Lorentzian manifold traced out by an "observer" (all tangent vectors are timelike).

More Definitions

Minkowski space $\mathbf{R}_1^n$ (like Euclidean space in the Riemannian case) is the model space for flat Lorentzian manifolds. The model spaces for constant positive and negative curvature Lorentzian manifolds are called *de Sitter* and *anti-de Sitter space* respectively:

$$\mathbf{S}_1^n = \{\mathbf{v} \in \mathbf{R}_1^{n+1} \mid \langle \mathbf{v}, \mathbf{v} \rangle = +1\}$$

and

$$\mathbf{H}_1^n = \{\mathbf{v} \in \mathbf{R}_2^{n+1} \mid \langle \mathbf{v}, \mathbf{v} \rangle = -1\}$$

Note that the definitions (and the notation) are completely analogous to the Riemannian model spaces. Here's a picture of de Sitter space – it is sometimes used as a model of an "inflationary universe":

Main Problems

A *spacetime* will be a compact Lorentzian manifold with non-empty spacelike boundary (compactness is essential for what I'll describe, though really only of the spacelike slices so $M \times \mathbf{R}$ with M closed is fine). We will assume it is *time-orientable* so we can talk about the *past boundary* and the *future boundary*:

I. Determine all possible topologies of the universe (i.e. of "spacelike slices").

II. More generally, describe the moduli space of Lorentzian metrics on a fixed topological type $M^3 \times \mathbf{R}$ (or $M^2 \times \mathbf{R}$ as a non-trivial "warmup").

III. Can the topology of the universe change?

Remarks On These Problems

I. We leave the determination of the topology of the *actual* universe to the physicists; N. Cornish, J. Weeks, et al suggest that the universe ought to be a small-volume closed hyperbolic 3-manifold (the "circles in the sky" business). Later in the talk we'll see some non-trivial topological constraints on spacelike slices.

II. Here's a more precise and tractable version: describe the moduli space $\Lambda(M)$ of *constant curvature* Lorentzian metrics on $M \times \mathbf{R}$ which are *causally trivial* (every world line crosses $M \times \frac{1}{2}$ exactly once) and maximal such.

Motivation: In addition to being physically interesting, it turns out that the Lorentzian moduli space can often be reduced to studying various (well-understood or else interesting) Riemannian moduli problems (e.g. something like Teichmüller theory when M is a surface). Information flows in both directions.

III. We will see that in many cases there is a negative answer. These are called, cleverly enough, "no topology change" theorems. They fall out from the analysis of **II**.

Some flat examples

A hyperbolic metric on M^n is given by a cocompact lattice $\pi_1(M) \cong \Gamma \subset O(n,1)$; this is the subgroup of isometries of $\mathbf{R}_1^{n+1}$ fixing $\mathbf{0}$; the quotient of the interior of the upper cone defines a flat Lorentzian metric on $M \times \mathbf{R}$.

This construction provides a Teichmüller space's worth of flat spacetimes when $n = 2$; a single example for $n \geq 3$ by Mostow rigidity.

Causal horizons

These examples are causally trivial in the sense defined earlier and also maximal: if we define the *causal horizon* of a spacetime to be the boundary of a maximal causally trivial extension, then it is easy to show that points appear in the causal horizon precisely when there is a lightlike straight line missing a slice $M \times \{t\}$. Thus for these examples the causal horizon is the cone.

Determining the action of the holonomy group Γ on the causal horizon is often fun and physically interesting: in particular this is how we prove "no-topology-change" theorems. The point is that for a topology-change, a spacetime has to evolve past its causal horizon (due to Geroch). For the examples just given, this is impossible: the action of Γ fails to be discontinuous when we reach the cone (which is identified with the space of horospheres – action is minimal/ergodic by work of Hedlund, Veech, Ratner, etc.).

Are there other examples?

A flat Lorentzian metric in $\Lambda(M)$ defines a homomorphism $\rho : \pi_1(M) \to Isom(\mathbf{R}_1^{n+1})$. Let

$$L : Isom(\mathbf{R}_1^{n+1}) \to O(n,1)$$

take an isometry $x \mapsto Ax + b$ to its "linear part" A. Then

$$\rho(\gamma)x = L(\rho(\gamma))x + t_\gamma$$

where $t : \pi_1(M) \to \mathbf{R}_1^{n+1}$ is a 1-cocycle called the *translational part* of ρ. The cocycle condition means that for every $\alpha, \beta \in \pi_1(M)$ we have

$$t_{\alpha\beta} = t_\alpha + L(\rho(\alpha))t_\beta.$$

The examples from the previous slide are those for which L maps $\pi_1(M)$ isomorphically to the lattice Γ and $t_\gamma = 0$ for all $\gamma \in \pi_1(M)$. With a little more work one gets:

Prop'n If $\dim H^1(\Gamma, \mathbf{R}_1^{n+1}) = k$ then we get a k-dimensional family of spacetimes embedded in $\Lambda(M)$ whose holonomy groups have non-trivial translational parts.

Cohomology Calculations, part 1

Group cohomology of an $SO(n,1)$ lattice Γ with $\mathbf{R}_1^{n+1}$ coefficients arises in a related context: the study deformations of Γ in $SO(n+1,1)$. For a familiar dimension like $n = 2$ we are talking about deforming a Fuchsian group in $SO(2,1)$ into quasi-Fuchsian groups in $SO(3,1)$.

You can think of a tangent vector to a curve of representations as an assignment $c : \Gamma \to so(n+1,1)$ of a Lie algebra element to each element of Γ telling it "which direction to go". This assignment needs to transform in the correct way:

$$c(\alpha\beta) = c(\alpha) + \alpha c(\beta)$$

which is the cocycle condition again. Coboundaries correspond to curves of representations gotten by conjugation.

Upshot: first-order deformations of Γ into $SO(n+1,1)$ are given by cohomology classes in $H^1(\Gamma, so(n+1,1))$.

Recall that a *flat conformal structure* on M^n is defined by charts into $\mathbf{S}^n$ with transition functions in $Mob^+(\mathbf{S}^n) = SO(n+1,1)^0$. Thus these cohomology classes also represent first order deformations of M's flat conformal structure (by the holonomy theorem).

Cohomology Calculations, part 2

Where does $\mathbf{R}_1^{n+1}$ come in? As a Γ-module, the Lie algebra splits:

$$so(n+1,1) \cong so(n,1) \oplus \mathbf{R}_1^{n+1}$$

and therefore so does the cohomology

$$H^1(\Gamma, so(n+1,1)) \cong H^1(\Gamma, so(n,1)) \oplus H^1(\Gamma, \mathbf{R}_1^{n+1}).$$

For $n = 2$, each of the terms on the right hand side is $6g-6$ dimensional where g is the genus of M^2 (the first piece is tangent to Teichmüller space). It turns out that the constructions given earlier give all flat $(2+1)$-spacetimes and so the moduli space is exactly $12g-12$ dimensional. Later we will give a geometric parameterization of this space in terms of Teichmüller space $Teich$ and measured lamination space ML.

Cohomology Calculations, part 3

For closed hyperbolic n-manifolds M^n, $n \geq 3$, we have from Mostow rigidity (or local rigidity - Calabi/Weil)

$$H^1(\Gamma, so(n,1)) = 0$$

and so

$$H^1(\Gamma, so(n+1,1)) \cong H^1(\Gamma, \mathbf{R}^{n+1}_1).$$

Upshot: We can deform the flat spacetime $M \times \mathbf{R}$ with holonomy Γ if and only if we can deform Γ *to first order* in $SO(n+1,1)$ if and only if M's flat conformal structure deforms *to first order*.

Bends and non-bends

Most known constructions of deformations come from "bending" along an embedded, totally geodesic, co-dimension-one surface or some variant thereof. Many people have contributed to this: Thurston, Apanasov, Johnson-Millson, Kourouniotis, Kapovich, Tan. The first "non-bends" were found by Misha Kapovich and come from reflection groups.

Here are some other examples of non-bends: the n-fold cyclic branched covers of the figure-eight knot $n \geq 4$ (the "Fibonacci manifolds") satisfy $H^1(\Gamma, \mathbf{R}^4_1) \neq 0$. These manifolds have zero first Betti number; in fact the 4-fold cyclic branched cover is non-Haken and contains no immersed totally geodesic surfaces.

Conversely, Misha also gave the first examples of closed hyperbolic 3-manifolds admitting no deformations in $SO(4,1)$. They are obtained by Dehn filling on hyperbolic two-bridge knots; I have recently extended his argument to work for surgeries on a wider class of cusped 3-hyperbolic manifolds.

Some answers, de Sitter case

Question **II** is answered in the de Sitter case by the following (all dimensions in fact):

Theorem The moduli space of de Sitter spacetimes $\Lambda(M)$ is parameterized by flat conformal structures on M.

When M is a surface, a flat conformal structure is the same thing as a complex projective structure since $\mathbf{S}^2 = \mathbf{C}P^1$ and the (orientation-preserving) Möbius group is $PSL(2, \mathbf{C})$. In particular, all surfaces admit such structures and a lot is known about the deformation spaces – there is a complex analytic parameterization via holomorphic quadratic differentials and also a geometric parameterization due to Thurston in terms of *grafting* (my other talk).

In higher dimensions, there are topological obstructions to having a flat conformal structure. For instance, Nil or $Solv$ torus bundles over S^1 do not admit such structures. It is not known exactly which 3-manifolds admit Möbius structures, even assuming the geometrization conjecture (though much is known due to Goldman, Kapovich, Luo, et al).

Some answers, $2+1$ case in general

For simplicity I will focus on the flat and de Sitter cases. If M is a hyperbolic surface, it turns out that up to a time reversal, a maximal spacetime $M \times \mathbf{R}$ is past complete but future incomplete. The future causal horizon has the structure of an $\mathbf{R}$-tree dual to some measured lamination; the spacelike slices converge to a point of $Teich$ in the past. The main result is that spacetimes are parameterized by this data at past and future infinity (the flat case was done originally by Geoff Mess in 1990):

Theorem Let M^2 be a closed hyperbolic surface. In both the flat and de Sitter cases, $\Lambda(M)$ is parameterized (in this special way) by $Teich \times ML$.

Corollary (no topology change) Any 3-dimensional spacetime of constant curvature is homeomorphic to $M^2 \times [0,1]$.

Idea of the proof of Corollary: The theorem describes precisely the structure of the causal horizon, and as before, it suffices to show that the spacetime cannot evolve past this horizon, or equivalently that the holonomy ceases to be discontinuous (to be honest, it does evolve past the horizon in some trivial cases, but without ruining the product structure).

$3 + 1$-dimensions

The key step in the $2+1$-dimensional case was finding the $\mathbf{R}$-tree in the future causal horizon. In the $3+1$-dimensional case, little is known about the structure of this horizon when it is not an $\mathbf{R}$-tree.

Open question What are the possibilities for the structure of the causal horizon when M is a closed hyperbolic 3-manifold and $M \times \mathbf{R}$ comes from a class in $H^1(M, \mathbf{R}^4_1)$? The case of the Fibonacci manifolds mentioned earlier is particularly mysterious to me.

At least there is an answer to question $\mathbf{I}$ for the case of flat $(3+1)$-spacetimes:

Theorem M^3 is a spacelike slice of a flat spacetime if and only if M^3 is modelled on $\mathbf{H}^3$, $\mathbf{E}^3$, or $\mathbf{H}^2 \times \mathbf{R}$.