Generalized bending laminations for hyperbolic n-manifolds

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Let $M = \Gamma\backslash\mathbb{H}^n$ be a closed oriented hyperbolic n-manifold. It will be useful in all that follows to think of Γ as a lattice in $SO(n,1)$.

There are two related deformation problems (in hyperbolic and Lorentzian geometry, respectively) which are controlled by the group cohomology $H^1(\Gamma, \mathbf{R}^{n+1})$ with coefficients in the standard representation of $SO(n,1)$ on Minkowski space.

(One can think of the group cocycles as functions $c : \Gamma \to \mathbf{R}^{n+1}$ satisfying the cocycle relation

$$c(ab) = c(a) + a \cdot c(b).$$

The coboundaries are of the form

$$c(a) = v - av.)$$

The first context in which this cohomology group arises is the study deformations of Γ in $SO(n+1,1)$.

When $n = 2$, for example, we are talking about deforming a Fuchsian group from $SO(2,1)$ into a quasi-Fuchsian group in $SO(3,1)$.

A tangent vector to a curve of representations amounts to an assignment $c : \Gamma \to \mathfrak{so}(n+1,1)$ of a Lie algebra element to each element of Γ. The condition that the relations in the group hold to first order is precisely the cocycle condition, and the coboundaries correspond to curves of representations obtained by conjugation.

Thus $H^1(\Gamma, \mathfrak{so}(n+1,1))$ can be thought of as the space of first-order deformations of Γ in $SO(n+1,1)$, up to conjugation.

Now as a Γ-module, the Lie algebra splits:

$$\mathfrak{so}(n+1,1) \cong \mathfrak{so}(n,1) \oplus \mathbf{R}^{n+1}$$

and therefore so does the cohomology

$$H^1(\Gamma, \mathfrak{so}(n+1,1)) \cong H^1(\Gamma, \mathfrak{so}(n,1)) \oplus H^1(\Gamma, \mathbf{R}^{n+1}).$$

so

$$H^1(\Gamma, \mathfrak{so}(n+1,1)) \cong H^1(\Gamma, \mathbf{R}^{n+1})$$

when $n > 2$ by (local) rigidity.

If $n = 2$, then this is a splitting of the tangent space to quasi-Fuchsian space into the Teichmüller directions and the "other" directions.

Before discussing the second deformation problem, let's look at what's known about the dimension of $H^1(\Gamma, \mathbf{R}^{n+1})$ in general.

When $n = 2$, the dimension is easily computed and equals the dimension of the Teichmüller space of M.

For $n \geq 3$, very little is known in general.

An embedded totally geodesic submanifold yields a bending deformation; the tangent vector to such a deformation is a non-zero class.

In joint work with Anneke Bart, we've shown that you get cohomology even if the surface is immersed, but "nearly embedded" (the figure-eight knot complement shows that some assumption is essential). And more generally for the "branched totally geodesic surfaces" of Kapovich-Millson satisfying a similar condition.

Not all examples come from branched totally geodesic surfaces.

The k-fold cyclic branched covers of the figure-eight knot $k \geq 4$ (the "hyperbolic Fibonacci manifolds") satisfy $\dim H^1(\Gamma, \mathbf{R}^4) = 2$. These manifolds have zero first Betti number; in fact the 4-fold cyclic branched cover is non-Haken and contains no branched totally geodesic surfaces.

These examples provide the main motivation for this work, in the following sense . . .

Other than the original bending construction, none of the constructions of cohomology classes is known to produce integrable deformations (despite isolated examples of Apanasov: his "stamping" deformations, etc.)

Indeed a central open question in this area is whether or not the representation variety of Γ in $SO(n+1,1)$ is smooth at the inclusion.

Kapovich has conjectured that it is and gave a proof for cocompact reflection groups.

The goal of this talk is to give a construction which assigns to a cohomology class a geometric object in the manifold which "explains" the class and which (hopefully) can be used to attack the integrability question.

The construction works by recasting in terms of the second deformation problem mentioned on the first slide. Namely, we will now view the space $H^1(\Gamma, \mathbf{R^{n+1}})$ as the space of affine deformations of the linear action of Γ on Minkowski space:

Since $Isom(\mathbf{R}^{n+1}) \cong O(n,1) \ltimes \mathbf{R}^{n+1}$, a representation $\rho : \pi_1(M) \to Isom(\mathbf{R}^{n+1})$ may be decomposed into its linear and translational parts:

If $L : Isom(\mathbf{R}^{n+1}) \to O(n,1)$ takes an isometry $x \mapsto Ax + b$ to its "linear part" A, then

$$\rho(\gamma)x = L(\rho(\gamma))x + t_\gamma$$

where $t : \pi_1(M) \to \mathbf{R}^{n+1}$ is a 1-cocycle;

$$t_{\alpha\beta} = t_\alpha + L(\rho(\alpha))t_\beta.$$

Construction, step one.

The group Γ acts discretely in the interior of the future light cone, with quotient spacetime $M \times \mathbf{R}$ (pictured for the surface case $n = 2$):

Some language: hyperbolic space is "space-like", so should be thought of as a copy of the "universe" at some time. The interior of the light cone is the "domain of dependence" generated by this universe; the maximal spacetime in which all observers meet the fixed universe exactly once. The boundary is the "causal horizon".

A class $[c] \in H^1(\Gamma, \mathbf{R}^{n+1})$ gives an affine deformation of Γ into Γ'; we need a deformation of the spacetime as well.

This comes basically from the holonomy theorem; it is useful to make things explicit though: represent $[c]$ by a Γ-equivariant de Rham 1-form η on $\mathbb{H}^n$; write $\eta = d\xi$, where ξ is a vector-valued function which can be used to deform the inclusion of $\mathbb{H}^n$ into $\mathbf{R}^{n+1}$.

The compactness of M guarantees that if η is small enough the deformed copy of hyperbolic space remains spacelike. For large cohomology classes, scale down, deform as above, and scale back up.

The "crooked plane" examples of Margulis, Drumm, and Goldman show that cusps are a problem in this construction. They deform the (free) group Γ without a corresponding deformation of the spacetime.

Construction, step two.

The deformed copy of hyperbolic space depends on the choice of 1-form, but it generates a new domain of dependence, which doesn't depend on choices (this is, in a sense, the main theorem to the physicists; an infinite-dimensional space of universes "washes out" to a finite-dimensional collection).

Construction, step three.

Basic Lorentzian geometry (Hawking-Ellis) tells you a lot about the structure of the causal horizon. Each point is contained in a (future) complete null ray. There is a well-defined "spacelike part", characterized e.g. as the set of endpoints of these null rays.

Theorem. The causal horizon is convex with null and spacelike support planes.

Recall that a spacelike plane defines a point in hyperbolic space by taking a unit timelike perpendicular.

Fix a point in the spacelike part. The set of spacelike support planes containing this point defines a subset of hyperbolic space which is one of the strata of our stratification.

It is worth returning to the simplest examples.

Advanced geometric knowledge allows the “right” geometric choice for bending; it is good fun to do the same with, say, a harmonic 1-form as constructed by Kudla-Millson and see how the evolution of the spacetime “pulls it tight”.

Hopefully this convinces you that running the construction for a class coming from "bending" reproduces the bending lamination. In the case $n = 2$, you can then show the map from $H^1(\Gamma, \mathbf{R}^3)$ to $\mathcal{ML}$ is one-one, giving a fun proof that $\mathcal{ML} \cong \mathbf{R}^{6g-6}$ (and in fact that it's a vector space; Mess did this in 1990).

For $n > 2$, one might hope to define a space of generalized bending laminations. There may be no universal way of recognizing these unfortunately; e.g. one gets depressed quickly thinking about which immersed totally geodesic surfaces support cohomology classes.