Cohomology constructions for hyperbolic knot and link complements

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The Problem

Let $M = \Gamma\backslash\mathbb{H}^3$ be a finite volume oriented hyperbolic 3-manifold, so Γ is a lattice in $SO(3,1)$.

The deformation theory of Γ within $SO(3,1)$ is well-known; if M has d cusps, then Γ has a $2d$ (real) dimensional character variety of deformations in $SO(3,1)$ near the inclusion.

We are interested in the local deformation theory of Γ in $SO(4,1)$. By the holonomy theorem this is equivalent to studying local deformations of the flat conformal (Möbius) structure on M, or deformations of the "quasi-Fuchsian" hyperbolic 4-manifold $\Gamma\backslash\mathbb{H}^4$, etc.

Infinitesimal Version

A tangent vector to a curve of representations amounts to an assignment $c : \Gamma \to \mathfrak{so}(4,1)$ of a Lie algebra element to each element of Γ. The condition that the relations in the group hold to first order is the cocycle condition

$$c(ab) = c(a) + a \cdot c(b)$$

and coboundaries correspond to curves of representations obtained by conjugation.

Therefore $H^1(\Gamma, \mathfrak{so}(4,1))$ can be thought of as the space of first-order deformations of Γ in $SO(4,1)$, up to conjugation.

When M has cusps, we define a subspace consisting of cocycles that are coboundaries when restricted to any cyclic subgroup generated by a parabolic (this is stronger than "parabolic preserving"). Passing to cohomology we write this subspace as $PH^1(\Gamma, \mathfrak{so}(4,1))$.

A Splitting

Now as a Γ-module, the Lie algebra splits:

$$\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathbf{R}^4$$

and therefore so does the cohomology

$$H^1(\Gamma, \mathfrak{so}(4,1)) \cong H^1(\Gamma, \mathfrak{so}(3,1)) \oplus H^1(\Gamma, \mathbf{R}^4).$$

By local rigidity,

$$H^1(\Gamma, \mathfrak{so}(4,1)) \cong H^1(\Gamma, \mathbf{R}^4)$$

in the closed case. When there are $d > 0$ cusps we have

$$PH^1(\Gamma, \mathfrak{so}(4,1)) \cong PH^1(\Gamma, \mathbf{R}^4)$$

and

$$\dim H^1(\Gamma, \mathfrak{so}(4,1)) = 2d + (\dim PH^1(\Gamma, \mathbf{R}^4) + d).$$

The $SO(4,1)$ "character variety"

The dimension just computed is the dimension of the character variety at the inclusion. Unlike the usual $SO(3,1)$ case however, this variety is *singular*.

The difference can be seen even looking at the restriction to a peripheral $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. In $SO(3,1)$ the centralizer of nearby representations is always two-dimensional (commuting parabolics or the stabilizer of a geodesic).

In $SO(4,1)$ the centralizer at the parabolic representation has one extra dimension, while at nearby elliptic/hyperbolic pairs the centralizer remains two-dimensional.

The figure-eight knot

One easy consequence of this singularity result is the following:

Theorem. If M has one cusp and $PH^1(\Gamma, \mathbf{R}^4) = 0$ then infinitely many Dehn fillings on M are locally rigid in $SO(4,1)$.

For example, a nice geometric argument of Misha Kapovich shows that any two-bridge knot or link complement satisfies $PH^1(\Gamma, \mathbf{R}^4) = 0$.

In particular, consider the figure-eight knot complement. We can do somewhat better in this case:

Theorem All but finitely many fillings on the figure-eight knot complement are are locally rigid in $SO(4,1)$.

A conjectural picture

What we've failed to understand so far is the the correct local structure of the character variety at the inclusion.

Bending classes

The main examples (and essentially only) examples of deformations in $SO(4,1)$ have been around for some time; these are the "bending deformations" coming from an embedded totally geodesic surface.

This works in any dimension: a codimension one surface splits the group as (say) an amalgamated product $A *_C B$, and the deformation can be obtained by conjugating A by the (one-dimensional, elliptic) centralizer of C.

In our infinitesimal computations we work with de Rham cohomology with coefficients in the flat $\mathfrak{so}(4,1)$ bundle (or $\mathbf{R}^4$ bundle) over M with holonomy Γ.

Bending classes can be represented by 1-forms dual to the bending surface, with values in $\mathbf{R}^4$ in the spacelike direction perp to the bending surface.

Generalized bends

We began by trying to find examples of cohomology classes supported on *immersed* totally geodesic surfaces, or more generally on "branched" totally geodesic surfaces of the kind considered by Apanasov and Kapovich-Millson.

The figure-eight knot shows that even in the presence of many immersed totally geodesic surfaces one can fail to get any cohomology.

On the other hand, we have the following lower bound:

Theorem. Suppose M contains a piecewise totally geodesic hypersurface with c_2 two dimensional components and c_1 one-dimensional components of the branch locus. Then the dimension of $(P)H^1(\Gamma, \mathbf{R}^4)$ is at least $c_2 - 2c_1$.

A partial converse

In earlier work I showed that a class in $H^1(\Gamma, \mathbf{R}^4)$ gives rise to what a so-called *generalized bending lamination* in M.

The idea is originally due to Geoff Mess: use the cohomology class to define an *affine* deformation of Γ in the isometry group of Minkowski space.

Then there is a canonical 1-form obtained by considering the equivariant map from $\mathbf{H}^3$ to the "causal horizon". If the original class was a bending deformation, this reproduces the bending hypersurface (hence the name).

An arithmetic example

In looking for concrete examples, we studied the Bianchi groups $\Gamma_d = PSL(2, O_d)$ and their finite index subgroups giving knot and links in S^3.

The idea is to look for deformation classes supported on subcomplexes of the *Mendoza complex*. This is a certain arithmetically-defined piecewise totally geodesic two-complex of the orbifold $\Gamma_d \backslash \mathbf{H}^3$.

It has been known for a while that the two-component link 8^2_{14} can be obtained as an index twelve subgroup of Γ_{-7}.

Supporting subcomplex

An older example

Not all examples come from branched totally geodesic surfaces.

The k-fold cyclic branched covers of the figure-eight knot $k \geq 4$ satisfy dim $H^1(\Gamma, \mathbf{R}^4) = 2$. These manifolds have zero first Betti number; in fact the 4-fold cyclic branched cover is non-Haken and contains no branched totally geodesic surfaces.

These manifolds are also two-fold cyclic branched covers of the Turk's head links (starting with 8_{18}), and similar arguments show that these knots and links satisfy dim $PH^1(\Gamma, \mathbf{R}^4) = 2$.

Spine of 8_{18}

The computations for the Fibonacci manifolds and Turk's head links offer no insight into what the generalized bending lamination looks like.

We performed computations similar to the ones for 8^2_{14}, by finding a piecewise totally geodesic spine for 8_{18} and computing representative classes supported on the spine. The results were much different.

It turns out that for this knot *no* representative of a class in PH^1 can be supported on a proper subcomplex, and further that vectors assigned to faces are sometimes timelike, unlike bending classes.