Deformation problems in hyperbolic and Lorentzian geometry

Kevin P. Scannell
Saint Louis University

Some Definitions

Minkowski space $\mathbf{R}^n_1$ is just $\mathbf{R}^n$ equipped with the standard signature $(n-1,1)$ inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + \ldots + v_{n-1} w_{n-1} - v_n w_n.$$

A *Lorentzian manifold* is defined like a Riemannian manifold, except the tangent spaces look like Minkowski space instead of Euclidean space. Lorentzian manifolds are models for general relativity:

$\mathbf{v}$ is *spacelike* if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$

$\mathbf{v}$ is *timelike* if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$

$\mathbf{v}$ is *lightlike* if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.

Main Questions

A *spacetime* will be a compact Lorentzian manifold with non-empty spacelike boundary.

I. What is the topology of the universe (i.e. the topology of a "spacelike slice")?

II. Describe the moduli space of Lorentzian metrics on a fixed topological type $M^3 \times [0,1]$ (or $M^2 \times [0,1]$ as a non-trivial "warmup").

III. Can the topology of the universe change?

Remarks On These Questions

I. Left to the physicists; e.g. N. Cornish and J. Weeks suggest that the universe ought to be a small-volume closed hyperbolic 3-manifold ("circles in the sky"). We assume all spacelike slices are closed hyperbolic (2- or) 3-manifolds.

Motivation: the geometry of a hyperbolic manifold gives substantial information about questions **II** and **III**; conversely (and more importantly) understanding the Lorentzian geometry of $M \times [0, 1]$ can give information about the geometry and topology of M.

II. Here's a more precise and tractable version: describe the moduli space $\Lambda(M)$ of *constant curvature* Lorentzian metrics on $M \times [0, 1]$ which are *causally trivial*: every world line crosses $M \times \frac{1}{2}$ exactly once.

The model space for flat Lorentzian manifolds is $\mathbf{R}^n_1$. The model spaces for constant positive and constant negative curvature are *de Sitter* and *anti-de Sitter* space respectively.

III. Answer follows from **II** (more later).

Some Examples

A hyperbolic metric on M^n is given by a cocompact lattice $\pi_1(M) \cong \Gamma \subset O(n,1)$; this is the subgroup of isometries of $\mathbf{R}_1^{n+1}$ fixing $\mathbf{0}$; the quotient of the interior of the upper cone defines a flat Lorentzian metric on $M \times \mathbf{R}$.

We get a Teichmüller space's worth of flat metrics when $n = 2$; a single example for $n \geq 3$ by Mostow rigidity.

Are there other examples?

A flat Lorentzian metric in $\Lambda(M)$ defines a homomorphism $\rho : \pi_1(M) \to Isom(\mathbf{R}_1^{n+1})$. Let $L : Isom(\mathbf{R}_1^{n+1}) \to O(n,1)$ take an isometry $Ax + b$ to its "linear part" A. Then

$$\rho(\gamma)x = L(\rho(\gamma))x + t_\gamma$$

where $t : \pi_1(M) \to \mathbf{R}_1^{n+1}$ is a 1-cocycle;

$$t_{\alpha\beta} = t_\alpha + L(\rho(\alpha))t_\beta.$$

The examples from the previous page are those for which $t_\gamma = 0$ for all $\gamma \in \pi_1(M)$. With a little more work one gets:

Theorem (Mess). "Yes" iff $H^1(M, \mathbf{R}_1^{n+1}) \neq 0$.

Cohomology Calculations

For $n = 2$, this cohomology group is easily computed; it is $6g - 6$ dimensional, where g is the genus of M^2.

For closed hyperbolic 3-manifolds M^3, very little is known. It was conjectured that the non-vanishing of this cohomology group was equivalent to the existence of a closed embedded quasi-Fuchsian surface in M; (unfortunately) this is false:

Theorem (S.) The n-fold cyclic branched covers of the figure-eight knot $n \geq 4$ (the "Fibonacci manifolds") satisfy $H^1(M, \mathbf{R}^{n+1}_1) \neq 0$.

These manifolds have zero first betti number; in fact the 4-fold cyclic branched cover is non-Haken.

Main Results for 2+1

A spacetime $M \times [0,1]$ equipped with a metric in $\Lambda(M)$ embeds in a *maximal spacetime* homeomorphic to $M \times \mathbf{R}$. It is more convenient to work with the moduli space of maximal spacetimes $\tilde{\Lambda}(M)$. Let $Teich(M)$ denote the Teichmüller space of a hyperbolic surface M, and $ML(M)$ the space of measured laminations. The following give answers to questions **II** and **III** respectively.

Theorem (Mess; S.) Let M^2 be a closed hyperbolic surface. In the flat and de Sitter cases, $\tilde{\Lambda}(M)$ is parameterized by $Teich(M) \times ML(M)$.

It turns out that up to a time reversal, every maximal spacetime is past complete but future incomplete. The future causal horizon has the structure of an $\mathbf{R}$-tree dual to some measured lamination; this is the second component in the parameterization. Other tools: Thurston's parameterization of projective structures and results on the "grafting map" of Teich(M) (joint with M. Wolf).

Theorem (Mess; S.) Any 3-dimensional spacetime of constant curvature is homeomorphic to $M^2 \times [0,1]$.