Deformation problems in hyperbolic and Lorentzian geometry

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Some Definitions

Minkowski space $\mathbb{R}^{n}_1$ is just $\mathbb{R}^n$ equipped with the standard signature $(n-1,1)$ inner product:

$$\langle v, w \rangle = v_1 w_1 + \ldots + v_{n-1} w_{n-1} - v_n w_n.$$ 

A Lorentzian manifold is defined like a Riemannian manifold, except the tangent spaces look like Minkowski space instead of Euclidean space. Lorentzian manifolds are models for general relativity:

$\mathbf{v}$ is spacelike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$

$\mathbf{v}$ is timelike if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$

$\mathbf{v}$ is lightlike if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$. 
Main Questions

A spacetime will be a compact Lorentzian manifold with non-empty spacelike boundary.

I. What is the topology of the universe (i.e. the topology of a “spacelike slice”)?

II. Describe the moduli space of Lorentzian metrics on a fixed topological type $M^3 \times [0, 1]$ (or $M^2 \times [0, 1]$ as a non-trivial “warmup”).

III. Can the topology of the universe change?
Remarks On These Questions

I. Left to the physicists; e.g. N. Cornish and J. Weeks suggest that the universe ought to be a small-volume closed hyperbolic 3-manifold (“circles in the sky”). We assume all spacelike slices are closed hyperbolic (2- or) 3-manifolds.

Motivation: the geometry of a hyperbolic manifold gives substantial information about questions II and III; conversely (and more importantly) understanding the Lorentzian geometry of $M \times [0, 1]$ can give information about the geometry and topology of $M$.

II. Here’s a more precise and tractable version: describe the moduli space $\Lambda(M)$ of constant curvature Lorentzian metrics on $M \times [0, 1]$ which are causally trivial: every world line crosses $M \times \frac{1}{2}$ exactly once.

The model space for flat Lorentzian manifolds is $R^n_1$. The model spaces for constant positive and constant negative curvature are de Sitter and anti-de Sitter space respectively.

III. Answer follows from II (more later).
Some Examples

A hyperbolic metric on $M^n$ is given by a cocompact lattice $\pi_1(M) \cong \Gamma \subset O(n,1)$; this is the subgroup of isometries of $\mathbb{R}^{n+1}_1$ fixing $0$; the quotient of the interior of the upper cone defines a flat Lorentzian metric on $M \times \mathbb{R}$.

We get a Teichmüller space’s worth of flat metrics when $n = 2$; a single example for $n \geq 3$ by Mostow rigidity.
Are there other examples?

A flat Lorentzian metric in $\Lambda(M)$ defines a homomorphism $\rho : \pi_1(M) \to Isom(\mathbb{R}^{n+1})$. Let $L : Isom(\mathbb{R}^{n+1}) \to O(n,1)$ take an isometry $Ax + b$ to its “linear part” $A$. Then

$$\rho(\gamma)x = L(\rho(\gamma))x + t_\gamma$$

where $t : \pi_1(M) \to \mathbb{R}^{n+1}$ is a 1-cocycle;

$$t_{\alpha\beta} = t_\alpha + L(\rho(\alpha))t_\beta.$$ 

The examples from the previous page are those for which $t_\gamma = 0$ for all $\gamma \in \pi_1(M)$. With a little more work one gets:

**Theorem** (Mess). “Yes” iff $H^1(M, \mathbb{R}^{n+1}) \neq 0$. 
Cohomology Calculations

For \( n = 2 \), this cohomology group is easily computed; it is \( 6g - 6 \) dimensional, where \( g \) is the genus of \( M^2 \).

For closed hyperbolic 3-manifolds \( M^3 \), very little is known. It was conjectured that the non-vanishing of this cohomology group was equivalent to the existence of a closed embedded quasi-Fuchsian surface in \( M \); (unfortunately) this is false:

**Theorem** (S.) The \( n \)-fold cyclic branched covers of the figure-eight knot \( n \geq 4 \) (the “Fibonacci manifolds”) satisfy \( H^1(M, \mathbb{R}^{n+1}_1) \neq 0 \).

These manifolds have zero first betti number; in fact the 4-fold cyclic branched cover is non-Haken.
Main Results for 2+1

A spacetime $M \times [0, 1]$ equipped with a metric in $\Lambda(M)$ embeds in a maximal spacetime homeomorphic to $M \times \mathbb{R}$. It is more convenient to work with the moduli space of maximal spacetimes $\tilde{\Lambda}(M)$. Let $Teich(M)$ denote the Teichmüller space of a hyperbolic surface $M$, and $ML(M)$ the space of measured laminations. The following give answers to questions II and III respectively.

**Theorem** (Mess; S.) Let $M^2$ be a closed hyperbolic surface. In the flat and de Sitter cases, $\tilde{\Lambda}(M)$ is parameterized by $Teich(M) \times ML(M)$.

It turns out that up to a time reversal, every maximal spacetime is past complete but future incomplete. The future causal horizon has the structure of an $\mathbb{R}$-tree dual to some measured lamination; this is the second component in the parameterization. Other tools: Thurston’s parameterization of projective structures and results on the “grafting map” of $Teich(M)$ (joint with M. Wolf).

**Theorem** (Mess; S.) Any 3-dimensional spacetime of constant curvature is homeomorphic to $M^2 \times [0, 1]$. 