THE GENERALIZED CUSPIDAL COHOMOLOGY PROBLEM

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Abstract. Let $\Gamma \subset SO(3,1)$ be a lattice. The well-known *bending deformations*, introduced by Thurston and Apanasov, can be used to construct non-trivial curves of representations of Γ into $SO(4,1)$ when $\Gamma\backslash\mathbb{H}^3$ contains an embedded totally geodesic surface. A tangent vector to such a curve is given by a non-zero group cohomology class in $H^1(\Gamma, \mathbb{R}^4_1)$. Our main result generalizes this construction of cohomology to the context of "branched" totally geodesic surfaces. We also consider a natural generalization of the famous cuspidal cohomology problem for the Bianchi groups (to coefficients in non-trivial representations), and perform calculations in a finite range. These calculations lead directly to an interesting example of a link complement in S^3 which is not infinitesimally rigid in $SO(4,1)$. The first order deformations of this link complement are supported on a piecewise totally geodesic 2-complex.

1. Introduction

1.1. **Bianchi Groups.** Let d be a negative square-free integer and denote by $\mathfrak{O}_d$ the ring of integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$. The *Bianchi groups* $\Gamma_d = PSL(2, \mathfrak{O}_d)$ are discrete subgroups of $PSL(2,\mathbb{C})$, studied as far back as the 1890's by Picard and Bianchi (see for instance [6]). In the last 100 years a wide array of techniques have come into play in studying these groups and the corresponding 3-orbifolds, from number theory, geometric group theory, spectral geometry, and 3-manifold topology.

Our interest in these groups comes primarily from studying the topology of hyperbolic 3-manifolds, as many famous and beautiful examples of cusped hyperbolic 3-manifolds coming from finite index subgroups of the Bianchi groups.

For a variety of reasons it is a fundamental question to study the cohomology $H^*(\Gamma, \mathbb{R})$ of arithmetically-defined subgroups Γ of $PSL(2,\mathbb{C})$ (or of the isometry group of higher-dimensional hyperbolic spaces). Since $\mathbb{H}^n$ is contractible, we have

$$H^*(\Gamma, \mathbb{R}) \cong H^*(\Gamma\backslash\mathbb{H}^n, \mathbb{R})$$

and one can use topological and analytic techniques to compute cohomology (e.g. Hodge theory if $\Gamma\backslash\mathbb{H}^n$ is compact). The question becomes a bit more interesting when $\Gamma\backslash\mathbb{H}^n$ is not compact. In the 3-dimensional case, one observes that $\Gamma\backslash\mathbb{H}^3$ is the interior of a compact 3-manifold with boundary M, in which case the computation of, say, $H^1(M,\mathbb{R})$ can be performed in terms of the kernel and image of the restriction $H^1(M,\mathbb{R}) \to H^1(\partial M, \mathbb{R})$. Since the dimension of the image is computable using duality, we focus on the kernel of restriction; this subgroup can be identified with the cohomology of $\Gamma\backslash\mathbb{H}^3$ with compact supports, the space of *harmonic cusp forms* [20], or the *parabolic cohomology* $PH^1(\Gamma,\mathbb{R})$ (defined in §2). If the space of harmonic cusp forms vanishes, we say Γ has *vanishing cuspidal cohomology.* The main achievement in this area is the following result:

Theorem 1.1. *The Bianchi group Γ_d has vanishing cuspidal cohomology if and only if*

$$d \in \{-1, -2, -3, -5, -6, -7, -11, -15, -19, -23, -31, -39, -47, -71\}.$$

Certain vanishing results for small d use explicit presentations for Γ_d (some of which can be found in [2], [34], [38]) while the result above was made definitive in [41] using the so-called *Mendoza complex* [31].

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The non-vanishing results can be found in [5], [17], [35], and [43]. The Lefschetz fixed-point technique developed in Harder's papers [19], [20] was also critical in this development.

1.2. **Generalized Cuspidal Cohomology.** In this paper we will consider the generalization of the cuspidal cohomology problem where the trivial coefficients are replaced by some non-trivial finite-dimensional representation of $SO(3,1)$. Our choice of coefficients is guided in part by Raghunathan's vanishing theorem [7], [18], [32], [33]: For any discrete cocompact subgroup Γ of a connected simple Lie group G and any irreducible, finite-dimensional real representation V of G, we have $H^1(\Gamma, V) = 0$ unless G is locally isomorphic to $SO(n,1)$ or $SU(n,1)$. In fact, if G is locally isomorphic to $SO(n,1)$ then $H^1(\Gamma, V) = 0$ unless $V = \mathfrak{H}^j V_0$ where V_0 is the standard representation on Minkowski space $\mathbb{R}^{n+1}_1$ and $\mathfrak{H}^j V_0$ denotes the space of harmonic polynomials on V_0 of degree j. In this paper, we will consider coefficients in the standard representation $\mathbb{R}^4_1$; we do so in part because this case represents the simplest unresolved case in light of Raghunathan, but more significantly because of an important geometric interpretation of $PH^1(\Gamma, \mathbb{R}^4_1)$ in terms of bending deformations and their generalizations. We will provide the details of this connection in §2 and §3, remarking for now that most known constructions of non-zero classes in $PH^1(\Gamma, \mathbb{R}^4_1)$ come from the presence of an embedded totally geodesic surface in the 3-orbifold $\Gamma\backslash\mathbb{H}^3$. Since the Bianchi orbifolds (and the arithmetic manifolds commensurable with them) contain infinitely many immersed totally geodesic surfaces [28] we expect computations like the ones in this paper will provide a fruitful testing ground for any conjectural picture of the deformation theory of $SO(3,1)$ lattices.

1.3. **Main Results.** Our main results can be summarized as follows. In §3, we use the spectral sequence associated to a Γ-complex to give a generalization of the bending construction to "branched" totally geodesic hypersurfaces (including immersed surfaces); see Theorem 3.1. We also obtain, in §4, some partial results on the generalized cuspidal cohomology problem by making calculations in a finite range, showing in particular that $PH^1(\Gamma_d, \mathbb{R}^4_1) = 0$ for $d = -1, -2, -3, -7, -11, -15$. Using Theorem 3.1, this has the consequence that each of the (infinitely many) immersed closed totally geodesic surfaces in the above mentioned Bianchi orbifolds $\Gamma_d\backslash\mathbb{H}^3$ is "far from embedded" (see §3 for the precise definitions). Finally, we give an interesting example of a link complement in S^3 such that $PH^1(\Gamma, \mathbb{R}^4_1) \neq 0$, with non-zero classes supported on a piecewise totally geodesic 2-complex. See §5.

2. Cuspidal cohomology and deformation theory

2.1. **Infinitesimal Deformations.** We will review in this section the connection between the cohomology group $PH^1(\Gamma, \mathbb{R}^4_1)$ and deformations of a lattice Γ in $SO(3,1)$ into $SO(4,1)$.

More generally, consider the inclusion $\rho_0 : \Gamma \hookrightarrow SO(n,1)$ of a lattice in $SO(n,1)$ and consider the space of representations $\mathrm{Hom}(\Gamma, SO(n,1))$. The structure of this space in a neighborhood of ρ_0 has been well-understood for some time; it is a consequence of the local rigidity results of Calabi [9] and Garland-Raghunathan [13] that a neighborhood consists of representations conjugate to ρ_0 whenever $n > 3$ (and in the cocompact case for $n = 3$). Mostow Rigidity strengthens this to a global result: any discrete, faithful representation $\rho \in \mathrm{Hom}(\Gamma, SO(n,1))$ is conjugate to ρ_0 (for $n \geq 3$). To obtain an interesting deformation theory, we can include $SO(n,1) \hookrightarrow SO(n+1,1)$ and study the representation variety $\mathrm{Hom}(\Gamma, SO(n+1,1))$ in a neighborhood of the inclusion ρ_0.

Given $\rho_0 \in \mathrm{Hom}(\Gamma, SO(4,1))$, a *deformation* of ρ_0 is a smooth curve $\rho_t : [0,\epsilon] \to \mathrm{Hom}(\Gamma, SO(4,1))$. The tangent vector to a deformation ρ_t at $t = 0$ can be described by assigning an element of the Lie algebra $\mathfrak{so}(4,1)$ to each element as follows

$$c(\gamma) = \dot{\rho}(\gamma)\rho(\gamma)^{-1}.$$

The fact that each ρ_t is a homomorphism differentiates to the condition that this function $c : \Gamma \to \mathfrak{so}(4,1)$ satisfies the *cocycle condition*

$$c(\gamma_1\gamma_2) = c(\gamma_1) + Ad(\gamma_1)c(\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$, hence $c \in Z^1(\Gamma, \mathfrak{so}(4,1))$ is a group cocycle, where the action on $\mathfrak{so}(4,1)$ comes from the adjoint representation. Not surprisingly, deformations coming from conjugation in $SO(4,1)$ produce tangent vectors in $B^1(\Gamma, \mathfrak{so}(4,1))$ and so it is sensible to think of the group cohomology $H^1(\Gamma, \mathfrak{so}(4,1))$ as the "space of infinitesimal deformations" of ρ_0 in $SO(4,1)$.

In what follows we will assume that $\rho_0 : \Gamma \hookrightarrow SO(3,1) \subset SO(4,1)$. When $\rho_0(\Gamma)$ contains parabolic elements, we will only be interested in infinitesimal deformations which are trivial on the parabolics. Thus we define

$$PZ^1(\Gamma, \mathfrak{so}(4,1)) = \{c \in Z^1(\Gamma, \mathfrak{so}(4,1)) \mid \text{ for parabolic } \gamma \in \Gamma, c(\gamma) \in \mathrm{im}(1-\gamma)\}$$

and define the *parabolic cohomology*

$$PH^1(\Gamma, \mathfrak{so}(4,1)) = PZ^1(\Gamma, \mathfrak{so}(4,1))/B^1(\Gamma, \mathfrak{so}(4,1)).$$

A critical observation for us is that the Lie algebra $\mathfrak{so}(4,1)$ splits when viewed as an $SO(3,1)$-module:

$$\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathbb{R}^4_1$$

inducing a splitting in cohomology

$$H^1(\Gamma, \mathfrak{so}(4,1)) \cong H^1(\Gamma, \mathfrak{so}(3,1)) \oplus H^1(\Gamma, \mathbb{R}^4_1)$$

and similarly in parabolic cohomology. It is typical in our setup for Γ to be a lattice in $SO(3,1)$ in which case $H^1(\Gamma, \mathfrak{so}(3,1)) = 0$ when Γ is cocompact [9], [42] and $PH^1(\Gamma, \mathfrak{so}(3,1)) = 0$ in the non-cocompact case by [13]. For trivial coefficients, the parabolic cohomology coincides with the kernel of the restriction map from $H^1(\Gamma\backslash\mathbb{H}^3) \to H^1(\partial(\Gamma\backslash\mathbb{H}^3))$; the second author [37] showed that the same is true for $PH^1(\Gamma, \mathbb{R}^4_1)$. Thus, when working on the generalized cuspidal cohomology problem as presented in the introduction, it suffices to compute this parabolic cohomology group. To summarize: the *cuspidal cohomology of* Γ *with coefficients in the standard representation* parameterizes *infinitesimal parabolic-preserving deformations of* Γ *into* $SO(4,1)$.

2.2. **Known Results.** There are only a couple of known vanishing results in this context. Kapovich [23] has observed that when Γ is a lattice in $SO(3,1)$ generated by two parabolic elements (equivalently when $\Gamma\backslash\mathbb{H}^3$ is the complement of a hyperbolic two-bridge link in $\mathbb{S}^3$ – see [1]) then $PH^1(\Gamma, \mathbb{R}^4_1) = 0$, in which case we say Γ is *locally rigid* in $SO(4,1)$. This result is discussed (and applied) in the context of the Bianchi groups in §4.1 below. For closed 3-manifolds, the strongest statement was given by the second author in [37], where it was shown that one can obtain infinitely many locally rigid examples by Dehn filling on a locally rigid one-cusped 3-manifold.

In terms of non-vanishing, the bending construction produces non-zero classes in $H^1(\Gamma, \mathbb{R}^4_1)$ in the presence of a closed, embedded, two-sided totally geodesic surface (readers unfamiliar with this construction are directed to §3 where a more general result is proved). There are a handful of other isolated examples where it has been shown that $H^1(\Gamma, \mathbb{R}^4_1) \neq 0$ (e.g. [3], [4], [29], [36], [39]) but nothing is known in general.

Potyagailo has asked when an *immersed*, non-embedded totally geodesic surface gives rise to an infinitesimal deformation into $SO(4,1)$ (as attributed in [24]). This was a major motivating question for the present paper, given the abundance of totally geodesic surfaces contained in the Bianchi orbifolds and their finite covers. We note first that, by the main result of [27], there always exist (probably large) finite covers which admit bending deformations. On the other hand, our computations in §4.2 show that there exist many immersed totally geodesic surfaces in the Bianchi orbifolds that do not give rise to elements of $PH^1(\Gamma, \mathbb{R}^4_1)$, and in fact the full cohomology group $H^1(\Gamma_{-3}, \mathbb{R}^4_1) = 0$ for $d = -3$. The infinitesimal deformations constructed in §5 are supported on a piecewise totally geodesic 2-complex in an 12-sheeted cover of $\Gamma_{-7}\backslash\mathbb{H}^3$ but appear not to be supported on a single closed totally geodesic surface.

Our broader study of $PH^1(\Gamma, \mathbb{R}^4_1)$ for knot and link complements in $\mathbb{S}^3$ was influenced also by a conjecture of Menasco and Reid [30] which states that no hyperbolic knot complement in $\mathbb{S}^3$ contains an embedded totally geodesic surface. This conjecture is not addressed in this paper in light of the fact that the figure-eight knot is the only knot appearing as a finite index subgroup of a Bianchi group (and, as we will explain below, $PH^1 = 0$ in this case).

3. The spectral sequence associated to a Γ-complex

3.1. **Preliminaries.** Let Γ be a finitely presented group which acts on a vector space V of characteristic zero.

Given a cellular action of Γ on a complex X, there is a spectral sequence [8, Ch. VII] which computes $H^1(\Gamma, V)$ in terms of the cohomology of the simplex stabilizers. Things are substantially simpler when the Γ-complex X is contractible, as it will be in our setup. In this case, the E_1 term is given by:

$$E_1^{pq} = \oplus H^q(\Gamma_\sigma, V)$$

where the direct sum is over a set of representative p-cells σ and Γ_σ is the stabilizer of σ. If τ is a face of σ, then the fact that the action is cellular means $\Gamma_\sigma \subseteq \Gamma_\tau$ and so there is a restriction map $H^*(\Gamma_\tau, V) \to H^*(\Gamma_\sigma, V)$. If $\sigma = \gamma\sigma_0$ for a representative cell σ_0, then $\Gamma_{\sigma_0} = \gamma\Gamma_\sigma\gamma^{-1}$ and we may follow the restriction map by the map induced by conjugation; this composition is essentially d^1.

3.2. **Branched totally geodesic surfaces.** We will now describe in more detail a technique for obtaining non-vanishing results from "nearly embedded" totally geodesic surfaces.

One way to describe the bending deformations along embedded totally geodesic surfaces is in terms of the free product with amalgamation or HNN decompositions which they induce. In other words, if a totally geodesic surface S decomposes a 3-manifold group Γ into $\Gamma_1 *_{\pi_1(S)} \Gamma_2$, then bending can be expressed as

$$\rho_t(\gamma) = \gamma \text{ for } \gamma \in \Gamma_1 \text{ and } \rho_t(\gamma) = \alpha_t\gamma{\alpha_t}^{-1} \text{ for } \gamma \in \Gamma_2$$

where α_t is a one-parameter family of rotations in $SO(4,1)$ commuting with the Fuchsian group $\pi_1(S)$. Johnson and Millson [22] work out this point of view in the more general case of a finite collection of embedded surfaces decomposing Γ as a *graph of groups*.

Infinitesimally, this amounts to an application of the spectral sequence of the previous section to the tree associated to the graph of groups decomposition. The sequence collapses to a long exact sequence of the form:

$$H^0(\Gamma, \mathbb{R}^4_1) \to \oplus H^0(\Gamma_\tau, \mathbb{R}^4_1) \to \oplus H^0(\Gamma_\sigma, \mathbb{R}^4_1) \to H^1(\Gamma, \mathbb{R}^4_1) \to \oplus H^1(\Gamma_\tau, \mathbb{R}^4_1) \to \dots$$

where the τ are representative vertices (hence the Γ_τ are fundamental groups of three-dimensional complementary regions) and the σ are representative edges (hence the Γ_σ are Fuchsian surface groups). The first two terms above are clearly zero; each H^0 in the next term is one-dimensional and when included into $H^1(\Gamma, \mathbb{R}^4_1)$ these are precisely the tangent vectors to the bending deformations associated to the given decomposition. In fact, if one is only interested in the deformation-theoretic properties coming from the given decomposition, it is convenient to *impose* the requirement that $H^1(\Gamma_\tau, \mathbb{R}^4_1) = 0$ for each vertex τ and define the resulting classes in $H^1(\Gamma, \mathbb{R}^4_1)$ to be *infinitesimal bending deformations*.

In the case of a family of (possibly intersecting) immersed totally geodesic surfaces, Γ is decomposed as a *complex of groups* [24]; and one can attempt to compute deformations and infinitesimal deformations of Γ in this way. Finding integrable deformations appears quite difficult (this is the content of Potyagailo's question) so we can begin by computing $H^1(\Gamma, \mathbb{R}^4_1)$ using such a decomposition.

Kapovich and Millson define a *branched totally geodesic surface* S in a hyperbolic 3-manifold $M = \Gamma\backslash\mathbb{H}^3$ to be a closed subset that is locally modeled on either a totally geodesic plane or the "binding" of a collection of (three or more) totally geodesic half-planes meeting in a geodesic segment. The set of points with neighborhoods of the latter type is the *branch locus* B. We follow the presentation of [24], though we allow, additionally, triple points coming from totally geodesic planes crossing the branch locus transversely.

Given such a surface, the complex of groups decomposition of Γ comes from an action of Γ on the abstract complex X "dual" to the universal cover $\tilde{S}$ in $\mathbb{H}^3$. The vertices of X correspond to the (3-dimensional) connected components of $\mathbb{H}^3 - \tilde{S}$; two vertices are joined by an edge when the corresponding subsets of $\mathbb{H}^3$ are separated by a component of $\tilde{S} - \tilde{B}$. The edges correspond to these components of $\tilde{S} - \tilde{B}$ and the faces correspond to the components of the branch locus $\tilde{B}$. Fortunately, since we are only interested in first cohomology, the 3-dimensional cells coming from triple points in $\tilde{S}$ can be ignored (indeed, the stabilizer of such a cell is finite and so the relevant column $E^1_{3,j} = \oplus H^j(\Gamma_\tau, \mathbb{R}^4_1)$ in the spectral sequence vanishes anyway).

The action of Γ on $\mathbb{H}^3$ induces an action on the strata and complementary regions of $\tilde{S}$ and hence on X. We assume (1) the normal bundles of the leaves have no holonomy which ensures the action has no edge inversions and (2) the fundamental groups of the 3-dimensional complementary regions are irreducible subgroups of $PSL(2, \mathbb{C})$. Our definitions imply that every edge σ has a (possibly elementary) Fuchsian stabilizer Γ_σ.

We have the following first order generalization of the bending construction:

Theorem 3.1. *Suppose $M = \Gamma\backslash\mathbb{H}^3$ is a complete hyperbolic 3-manifold containing a branched totally geodesic hypersurface S with branch locus B. Let c_2 be the number of two-dimensional complementary regions in $S - B$ (edges in the complex of groups) with non-elementary Fuchsian fundamental group, and let c_1 be the number of components of B (2-cells in the complex of groups). Then the space of infinitesimal bending deformations supported on S (a fortiori $H^1(\Gamma, \mathbb{R}^4_1)$) has dimension at least $c_2 - 2c_1$.*

Proof. In our setup the $E^1_{0,1}$ term is a sum $\bigoplus H^1(\Gamma_\tau, V)$ where the Γ_τ are the fundamental groups of the 3-dimensional complementary regions and, as above, the infinitesimal bending deformations form the subspace of $H^1(\Gamma, \mathbb{R}^4_1)$ computed by setting $E^1_{0,1} = 0$. Having declared this, any classes which survive in $E^2_{1,0}$ will survive to E^∞. But we get a lower bound on the dimension of this space by a trivial dimension count: By assumption, there are c_2 regions in $S-B$ with non-elementary Fuchsian fundamental group; each such group Γ_σ has a one-dimensional fixed set in $\mathbb{R}^4_1$ (the perp of the indefinite plane it leaves invariant) and so $\dim H^1(\Gamma_\sigma, \mathbb{R}^4_1) = 1$. Thus $\dim E^1_{1,0} \geq c_2$. But in the worst case, the fundamental group of each component of the branch locus is a purely hyperbolic element (with a two-dimensional fixed set in $\mathbb{R}^4_1$) in which case $\dim E^1_{2,0} = 2c_1$. Also by assumption, the vertex stabilizers have no invariant vectors in $\mathbb{R}^4_1$, so $\dim E^1_{0,0} = 0$. Taking cohomology, we find $\dim E^2_{1,0} \geq c_2 - 2c_1$ as desired. ☐

In the non-singular case ($c_1 = 0$), this is precisely the infinitesimal version of Johnson and Millson's graph of groups estimate. Shamelessly selecting the terminology to fit our result, we say S is *nearly embedded* when $2c_1 < c_2$.

The contrapositive will be relevant in §4.3, where we provide examples of Bianchi groups and subgroups satisfying $PH^1(\Gamma, \mathbb{R}^4_1) = 0$. In particular the corollary applies to the complements of the figure-eight knot, Whitehead link, and Borromean rings.

Corollary 3.2. *Let $M = \Gamma\backslash\mathbb{H}^3$ be a complete hyperbolic 3-manifold such that $PH^1(\Gamma, \mathbb{R}^4_1) = 0$. Then no immersed closed totally geodesic surface in M is nearly embedded.*

Proof. If a surface S were nearly embedded, then the previous theorem gives a non-zero class z in $H^1(\Gamma, \mathbb{R}^4_1)$ supported on S. Since S is compact, $z \in \ker(\text{res})$. But $PH^1(\Gamma, \mathbb{R}^4_1) = \ker(\text{res})$ by [37], contradicting the assumption that $PH^1 = 0$. ☐

It should be noted that the paper of Kapovich and Millson goes much farther in developing a *non-abelian first cohomology variety* by means of which one can analyze the harder integrability questions; see also [25], [26].

4. Computational techniques

4.1. **Geometric arguments.** Kapovich first observed in [23] that $PH^1(\Gamma, \mathbb{R}^4_1) = 0$ whenever Γ is the fundamental group of a two-bridge knot complement.

Proposition 4.1. *If Δ is a lattice subgroup of $SO(3,1)$ generated by two parabolics, then $PH^1(\Delta, \mathbb{R}^4_1) = 0$.*

Proof. Let α_0 and β_0 be the parabolic generators in $SO(3,1)$, and let $c \in PZ^1(\Delta, \mathbb{R}^4_1) \subseteq PZ^1(\Delta, \mathfrak{so}(4,1))$, using the splitting discussed in §2.1. Now $c(\alpha_0) = (1-\alpha_0)v_0$, and $c(\beta_0) = (1-\beta_0)w_0$ for some $v_0, w_0 \in \mathfrak{so}(4,1)$, and we can define two curves of parabolics $\alpha_t = \exp(tv_0)\alpha_0 \exp(tv_0)^{-1}$ and $\beta_t = \exp(tw_0)\beta_0 \exp(tw_0)^{-1}$ in $SO(4,1)$. The key geometric observation is that any pair of unipotents in $SO(4,1)$ leaves invariant a round $\mathbb{S}^2$ in $\mathbb{S}^3$; to see this, imagine that the first fixes the point at infinity of $\mathbb{S}^3$ and acts as translation along the x-axis (note that it is essential here that the element is unipotent and not merely parabolic). The orbit of the point at infinity under the second unipotent determines a straight line ℓ. The plane spanned by ℓ and the x-axis (union the point at infinity) is the desired 2-sphere. Thus the group generated by α_t and β_t is conjugate back into $SO(3,1)$. This implies that $[c] \in PH^1(\Delta, \mathfrak{so}(3,1))$ and hence that $[c] = 0$. □

Corollary 4.2. $PH^1(\Gamma_d, \mathbb{R}^4_1) = 0$ *for* $d = -1, -3, -7$.

Proof. Among the Bianchi groups, it is well-known that the groups Γ_d with $d = -1, -3, -7$ contain torsion-free finite-index subgroups corresponding to two-bridge link complements in $\mathbb{S}^3$ (it is also known that these are the only Γ_d with this property [1],[14]). By the proposition, these link complements have trivial parabolic cohomology and by a transfer argument we see that these three Bianchi groups must as well. □

Similarly we have:

Corollary 4.3. $PH^1(\Gamma_d, \mathbb{R}^4_1) = 0$ *for* $d = -2, -15$.

Proof. We use the existence of commensurable arithmetic groups generated by a parabolic and an elliptic (of orders 4 and 6 respectively, see [14]). We begin in the same way and then must argue that an arbitrary pair consisting of a finite-order elliptic and a unipotent in $SO(4,1)$ leaves invariant a round $\mathbb{S}^2$ in $\mathbb{S}^3$. Again conjugate so that the unipotent fixes infinity and translates along the x-axis. Let C be the center of the Euclidean circle in $\mathbb{S}^3$ fixed by the elliptic, and let ℓ be the line through C perpendicular to the plane containing the fixed circle. The plane spanned by ℓ and the line through C in the direction of the x-axis is left invariant and therefore defines the desired $\mathbb{S}^2$. Finally, since these groups are merely commensurable, we have checked via explicit computation of the transfer maps (using the generators given in [14]) that the intersections with Γ_{-2} and Γ_{-15} also have vanishing PH^1. □

4.2. **Fox calculus.** We will now recall a convenient computational formalism due to R. H. Fox [11, 12]; our presentation follows Goldman [15], who used this approach in the case that Γ is a closed surface group.

Let $\mathbb{F}^m$ denote the free group on m generators. A *derivation of the group ring* $\mathbb{Z}\mathbb{F}^m$ is an element $D \in Z^1(\mathbb{F}^m, \mathbb{Z}\mathbb{F}^m)$ which can be thought of as a $\mathbb{Z}$-linear map from $\mathbb{Z}\mathbb{F}^m$ to itself, satisfying

$$D(\alpha\beta) = \epsilon(\beta)D(\alpha) + \alpha D(\beta). \tag{1}$$

There are certain distinguished derivations $\partial_i = \frac{\partial}{\partial x_i}$ corresponding to the generators of $\mathbb{F}^m$; these are defined by $\partial_i x_j = \delta_{ij}$. The $\{\partial_i\}$ satisfy many of the formal properties of freshman calculus, including the "mean value theorem":

$$a - \epsilon(a) = \sum (\partial_i a)(x_i - 1) \tag{2}$$

for any $a \in \mathbb{Z}\mathbb{F}^m$.

The space of cocycles for a free group is quite simple, as the next proposition shows.

Proposition 4.4. *There is an isomorphism between V^m and $Z^1(\mathbb{F}^m, V)$ given by*

$$(v_1, \ldots, v_m) \mapsto (g \mapsto \sum_{i=1}^{m} (\partial_i g) v_i) \tag{3}$$

Proof. It trivial to check that the expression given in (3) defines a cocycle. On the other hand, an arbitrary cocycle $c \in Z^1(\mathbb{F}^m, V)$ satisfies

$$\begin{aligned} c(g) &= c(g-1) \\ &= c\left(\sum (\partial_i g)(x_i - 1)\right) \\ &= \sum (c(\partial_i g)\epsilon(x_i - 1) + (\partial_i g)c(x_i - 1)) \\ &= \sum (\partial_i g)c(x_i). \end{aligned} \tag{4}$$

for any $g \in \mathbb{F}^m$. This defines an inverse to the map in (3) by sending c to $(c(x_1), \ldots, c(x_m)) \in V^m$. □

Now if Γ has a finite presentation $\{x_1, \ldots, x_m \mid r_1, \ldots, r_p\}$, then $Z^1(\Gamma, V)$ is in one-one correspondence with

$$\{c \in Z^1(\mathbb{F}^m, V) \mid c(r_j) = 0 \text{ for } j = 1, \ldots, p\} \tag{5}$$

which in turn is isomorphic to

$$\{(v_1, \ldots, v_m) \in V^m \mid \sum_{i=1}^{m} (\partial_i r_j) v_i = 0 \text{ for } j = 1, \ldots, p\} \tag{6}$$

by Proposition 4.4. This is the form which will be most useful for computations. We call the matrix $F = (\partial_i r_j)$ the *Fox matrix* corresponding to the presentation.

Suppose, returning to our primary setup, that Γ is a lattice in $SO(3,1)$ generated by parabolics $x_1, \ldots, x_m$ (e.g. the fundamental group of a hyperbolic link complement as in §5 below). As above $Z^1(\Gamma, \mathbb{R}^4_1) = \ker F$; while in the case of parabolic cohomology, we have that $PZ^1(\Gamma, \mathbb{R}^4_1)$ consists of m-tuples of the form $((1 - x_1)v_1, \ldots, (1 - x_m)v_m) \in V^m$ in $\ker F$. In practice, we compute $\ker FP$ where P is the block diagonal matrix with 4 by 4 blocks $(1 - x_j)$ and then throw out coboundaries (of the form $(v, \ldots, v)$) and elements of $\ker P$.

Since we are only concerned with linear algebra in these Fox matrix calculations, we are free to choose a convenient basis for $\mathbb{R}^4_1$ tailored to the group at hand. Recalling that the representation of $SO(3,1)^0$ on $\mathbb{R}^4_1$ is equivalent to the action of $PSL(2,\mathbb{C})$ on the space of 2 by 2 Hermitian matrices $A \cdot Q = AQ\bar{A}^t$, we select the following Hermitian matrices as basis elements when working with Γ_d: $\begin{pmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $\omega = \frac{1+\sqrt{d}}{2}$. Having done so, all entries of the Fox matrix will be integers or half-integers, facilitating calculation with the modern electronic computer.

We have performed these calculations for several of the examples with explicit presentations in [38] and [10]. We restrict ourselves to a brief discussion of the results for $d = -1, -3$ and how they relate to our bending results from §3.

We know [38] that Γ_{-1} has the following presentation:

$$< T, U, L, A | A^2, L^2, (AL)^2, (TL)^2, (UL)^2, TUT^{-1}U^{-1}, (TA)^3, (UAL)^3 >$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Γ_{-3} has the following group presentation:

$$< T, U, L, A | A^2, L^3, (AL)^2, L^{-1}TLUT, L^{-1}ULT^{-1}, TUT^{-1}U^{-1}, (TA)^3, (UAL)^3 >$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$, $L = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$, and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

In each case, a direct computation as outlined above recovers the fact that $PH^1(\Gamma_d, \mathbb{R}^4_1) = 0$. More is true, however; it turns out that $H^1(\Gamma_{-3}, \mathbb{R}^4_1) = 0$, providing a strong negative answer to the question raised by Potyagailo in this case. We have looked for, but have not yet found, a concrete geometric argument along the lines of the previous subsection directly accounting for this rigidity.

We contrast this result with the observation that for $d \neq -1, -3$, the modular surface is embedded and two-sided (see [10]), hence supports a bending deformation; this shows that $\dim(H^1(\Gamma_d, \mathbb{R}^4_1)) \geq 1$. Of course, since the modular surface is cusped, these deformations are not trivial on parabolics and therefore do not define classes in PH^1. If $d = -1$ on the other hand, then the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ maps the plane P in the upper half space model lying over the real line onto itself, but reverses the normal bundle, making the modular surface one-sided. Similarly, if $d = -3$, then the matrix $T = \begin{pmatrix} \omega^2 & 0 \\ 0 & -\omega \end{pmatrix}$, maps P to a plane TP that intersects P. Hence the modular surface is immersed in this case, and does not support even a first order deformation.

The only new value of d for which the Fox calculus approach gave a vanishing result was $d = -11$ (again using the presentation from Swan). We summarize the computations of this and previous section in the following

Theorem 4.5. *$PH^1(\Gamma_d, \mathbb{R}^4_1) = 0$ for $d = -1, -2, -3, -7, -11, -15$. Thus, for these values of d, none of the infinitely many immersed closed totally geodesic surfaces in the orbifold $\Gamma_d \backslash \mathbb{H}^3$ is nearly embedded.*

4.3. **Mendoza complex.** For a hyperbolic 3-manifold $M = \Gamma\backslash\mathbb{H}^3$ with t toral boundary components, we will see below

$$\dim(PH^1(\Gamma, \mathbb{R}^4_1)) = \dim H_2(M, \mathbb{R}^4_1) - t$$

Note that this is different from the trivial coefficient case where

$$\dim(PH^1(\Gamma, \mathbb{R})) = \dim H_2(M, \mathbb{R}) - t + 1.$$

For completeness, we will recall how to obtain this result for trivial coefficients, and describe why we obtain a slightly different result for non-trivial coefficients.

Our starting point is the following well-known fact about the cohomology of 3-manifolds (see [37] for a proof):

Proposition 4.6. *Let M be a compact, oriented 3-manifold with fundamental group Γ, and let V be a vector space of characteristic zero acted upon by Γ. Assume ∂M consists of tori. Then*

$$\dim H^1(\Gamma, V) = \dim \ker \mathrm{res} + \dim H^0(\pi_1(\partial M), V).$$

where res *denotes the restriction on first cohomology.*

Now consider the long exact sequence in cohomology for the pair $(M, \partial M)$:

$$H^0(M,V) \to H^0(\partial M, V) \to H^1((M,\partial M),V) \to H^1(M,V) \to H^1(\partial M, V) \to$$
$$H^2((M,\partial M),V) \to H^2(M,V) \to H^2(\partial M,V) \to H^3((M,\partial M),V) \to 0$$

For brevity, write $x = \dim \ker \mathrm{res}$ in what follows. When $V = \mathbb{R}$, we have $\dim H^0(\pi_1(\partial M), \mathbb{R}) = t$ and so $\dim H^1 = x + t$. The sequence becomes:

$$\mathbb{R} \to \mathbb{R}^t \to \mathbb{R}^{x+t-1} \to \mathbb{R}^{x+t} \to \mathbb{R}^{2t} \to \mathbb{R}^{x+t} \to \mathbb{R}^{x+t-1} \to \mathbb{R}^t \to \mathbb{R} \to 0$$

In the case of $\mathbb{R}^4_1$ coefficients, it is still the case that $\dim H^0(\pi_1(\partial M), \mathbb{R}^4_1) = t$ (each parabolic $\mathbb{Z} \oplus \mathbb{Z}$ corresponding to a boundary component has a single fixed (null) vector in $\mathbb{R}^4_1$). The sequence becomes:

$$0 \to (\mathbb{R}^4_1)^t \to (\mathbb{R}^4_1)^{x+t} \to (\mathbb{R}^4_1)^{x+t} \to (\mathbb{R}^4_1)^{2t} \to (\mathbb{R}^4_1)^{x+t} \to (\mathbb{R}^4_1)^{x+t} \to (\mathbb{R}^4_1)^t \to 0 \to 0$$

Since $\dim H_2(M,V) = \dim H^1(M,\partial M)$ by duality, the assertions at the beginning of the section follow immediately.

It is illuminating to consider the map $H^0(\partial M, V) \to H^1((M,\partial M),V)$. Suppose that $\partial M = T^2$, i.e. the boundary consists of a single torus. A 2-cycle supported on ∂M comes from $H^0(M)$ with trivial coefficients, and obviously this class does not survive in $H^1((M,\partial M)) = H_2(M)$. On the other hand, a class in $H^0(\partial M, \mathbb{R}^4_1)$ comes from the null vector in $\mathbb{R}^4_1$ fixed by the parabolic $\mathbb{Z}\oplus\mathbb{Z}$ subgroup; since this vector is not invariant under all of $\pi_1(M)$ (in fact, no vector is), this class survives in $H^1((M,\partial M),\mathbb{R}^4_1)$. This accounts for the difference in dimension of the cuspidal cohomology with trivial and non-trivial coefficients.

Vogtmann [41] used the Mendoza Complex to compute the dimension of the cuspidal cohomology with rational coefficients. The Mendoza complex is also a very useful tool for computing cuspidal cohomology with twisted coefficients. We will give a short description of the relevant definitions and results. For a more complete description we refer the reader to the original paper by Vogtmann [41].

The *Siegel distance* between a point (z,r) in $\mathbb{H}^3$ and a cusp $\lambda \in \mathbb{Q}(\sqrt{d})$, where $\lambda = \frac{\alpha}{\beta}$, $\alpha, \beta \in \mathfrak{O}_d$ is given by

$$d((z,r),\lambda) = \frac{\|\beta z - \alpha\|^2 + \|\beta r\|^2}{rN < \alpha, \beta >}$$

Where $\|\|$ is the standard complex norm, and $N < \alpha, \beta >$ is the norm of the ideal of $\mathfrak{O}_d$ generated by α and β. Recall that the norm of an ideal can be computed in several different but equivalent ways. If $< \alpha, \beta >< \overline{\alpha}, \overline{\beta} >=< n >$, where $n \in \mathbb{Z}$ then we say $N < \alpha, \beta >= n$. Alternatively, we can think of $N < \alpha, \beta >$ as the index of the lattice generated by α and β in the lattice generated by 1 and ω.

Given a cusp λ, we define *the minimal incidence set* $H(\lambda)$ as follows:

$$H(\lambda) = \{(z,r) \mid d((z,r),\lambda) \le d((z,r),\mu) \text{ for all cusps } \mu \ne \lambda\}.$$

In other words, the minimal incidence set of a cusp λ is the closure of the set of points in $\mathbb{H}^3$ which are closer to λ than to any other cusp. The *Mendoza complex* X_d is given by

$$X_d = \bigcup_{\lambda\ne\mu} H(\lambda)\cap H(\mu) = \bigcup_{\lambda} \partial H(\lambda)$$

Mendoza [31] proved the following theorem:

Theorem 4.7. *(Mendoza)*

(1) *X_d is an $SL(2,\mathfrak{O}_d)$-invariant, two-dimensional CW-complex, with cellular $SL(2,\mathfrak{O}_d)$ action.*
(2) *X_d is a deformation retract of $\mathbb{H}^3$ by an $SL(2,\mathfrak{O}_d)$-invariant deformation retraction.*
(3) *$X_d/SL(2,\mathfrak{O}_d)$ is a finite CW-complex.*

In the next section, we actually perform the same construction for a torsion-free finite index subgroup of a Bianchi group, giving a Mendoza-like complex upon which the group acts freely. Since this is a manifold and not an orbifold, we may apply the duality results from earlier in this section to obtain $\dim PH^1$ directly from H^2 of the complex.

Example 1. The figure-eight knot complement has a fundamental domain consisting of two ideal tetrahedra. Straightforward computations show that, combinatorially, the Mendoza complex X coincides with a standard spine in each of the tetrahedra.

Having computed the explicit elements of $PSL(2,\mathbb{C})$ which identify the edges of X, it is fairly straightforward to compute the boundary matrix from $C_2(X,\mathbb{R}^4_1) \to C_1(X,\mathbb{R}^4_1)$. This yields $\dim H_2(X,\mathbb{R}^4_1) = 1$ and hence (as expected by the two parabolics argument above) $\dim PH^1(\Gamma,\mathbb{R}^4_1) = 0$. Recall then that Corollary 3.2 applies and there are no nearly embedded immersed totally geodesic surface in the figure-eight knot complement. This result mitigates in part some of the frustration felt by the authors after failed attempts at constructing explicit finite covers of Dehn fillings in which these surfaces lift to embeddings.

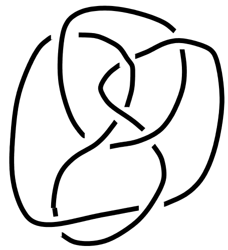

Figure 1. The link 8^2_{14}

5. A link complement in the three-sphere

A pleasant by-product of our calculations of PH^1 for the Bianchi groups was the discovery of a certain 2-component link in $\mathbb{S}^3$ with non-zero parabolic cohomology supported on a piecewise totally geodesic 2-complex analogous to the Mendoza complex. We should mention that in earlier unpublished work, the second author showed that PH^1 is two-dimensional for every Turk's head link – see the discussion in the final section of [36].

The finite index subgroups of Γ_{-7} have been studied in great detail by for instance Grunewald and Hirsch. It has been known for some time that there is a tessellation of $\mathbb{H}^3$ by ideal triangular prisms which is invariant under Γ_{-7}. We will be considering a certain index 12 torsion-free subgroup Γ of Γ_{-7} (denoted $\Gamma_{-7}(12, 21)$ in [16]) which has a pair of these prisms as a fundamental region, and such that $\Gamma\backslash\mathbb{H}^3$ is homeomorphic to the complement in $\mathbb{S}^3$ of the link 8^2_{14} depicted in figure 1. It had been shown previously by Hatcher that this link is commensurable with Γ_{-7} (see figure 15 of [21] and section 6.8 of [40]).

We have the following presentation of Γ from [16]:

$$\Gamma = \{x, y, z \mid xyx^{-1}y^{-1}zy^{-1}x^{-1}zxz^{-1}xzx^{-1}z^{-1}, xyx^{-1}y^{-1}zy^{-1}zx^{-1}z^{-1}y^{-1}zxz^{-1}yz^{-1}y\}.$$

We may choose x, y, and z so that $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ -2+2\omega & 1 \end{pmatrix}$, and $z = \begin{pmatrix} -1 & \omega \\ -\omega & \omega - 3 \end{pmatrix}$. We may once again choose the basis of §4.2 for $\mathbb{R}^4_1$: $\begin{pmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Doing so, the elements mentioned above are given by

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ \frac{-1}{2} & -1 & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}, y = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ -4 & -1 & -3 & 4 \\ -4 & -1 & 4 & 5 \end{pmatrix}, z = \begin{pmatrix} 5 & 1 & -2 & -4 \\ -4 & 0 & 1 & 3 \\ \frac{-7}{2} & 0 & \frac{5}{2} & \frac{7}{2} \\ \frac{-15}{2} & -1 & \frac{7}{2} & \frac{13}{2} \end{pmatrix}.$$

Direct computation shows that

$$\frac{\partial r_1}{\partial x} = 1 - xyx^{-1}(1 + y^{-1}zy^{-1}x^{-1}(1 - z(1 + xz^{-1}(1 - xzx^{-1})))),$$

$$\frac{\partial r_1}{\partial y} = x(1 - yx^{-1}y^{-1}(1 + zy^{-1})),$$

$$\frac{\partial r_1}{\partial z} = xyx^{-1}y^{-1}(1 + zy^{-1}x^{-1}(1 - zxz^{-1}(1 - x(1 - zx^{-1}z^{-1})))),$$

$$\frac{\partial r_2}{\partial x} = 1 - xyx^{-1}(1 + y^{-1}zy^{-1}zx^{-1}(1 - z^{-1}y^{-1}z))$$

$$\frac{\partial r_2}{\partial y} = x(1 - yx^{-1}y^{-1}(1 + zy^{-1}(1 + zx^{-1}z^{-1}y^{-1}(1 - zxz^{-1}(1 + yz^{-1})))))$$

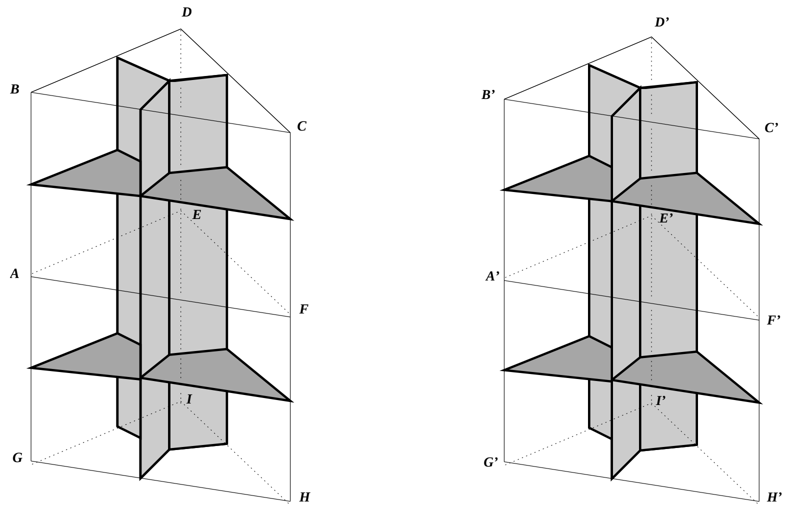

FIGURE 2. The Mendoza complex for $\mathbb{S}^3 \setminus 8^2_{14}$

$$\frac{\partial r_2}{\partial z} = xyx^{-1}y^{-1}(1 + zy^{-1}(1 - zx^{-1}z^{-1}(1 - y^{-1}(1 - zxz^{-1}(1 + yz^{-1}))))).$$

Unfortunately, z is not parabolic in this presentation, so it is convenient to rectify this with the following stratagem: If we introduce a new generator $z' = zxz^{-1}$, the first relation (rearranged a bit) implies that $z = yxy^{-1}x^{-1}z'x^{-1}(z')^{-1}xy$ so $\{x, y, z'\}$ is a generating set. Clearly z', as a conjugate of x, is parabolic (it is, in fact, $\begin{pmatrix} -\omega+1 & 1 \\ 2-\omega & \omega+1 \end{pmatrix}$).

The setup of §4.2 and a computer-assisted calculation of the Fox matrix shows then that $PZ^1(\Gamma, \mathbb{R}^4_1)$ is 5-dimensional. Since the generators have no common fixed vector in $\mathbb{R}^4_1$, B^1 is four-dimensional and we see $\dim PH^1(\Gamma, \mathbb{R}^4_1) = 1$. In fact a generating cocycle can be taken to be 0 on the generators x and y and vector $[0, 1, -2, 0]$ on z'.

We now discuss the same computation in light of the results of §4.3; it was shown there that to obtain the parabolic cohomology it suffices to compute $H^2(X, \mathbb{R}^4_1)$ for the Mendoza complex X. The goal here is to acquire some information about the supporting piecewise totally geodesic surface. We mentioned above that the fundamental domain for this link complement is given by two ideal double prisms as shown in figure 2.

Given that $\omega = \frac{1+\sqrt{-7}}{2}$, we have $A = 0,\ B = \frac{\omega}{2},\ C = \omega,\ D = \frac{\omega+1}{2},\ E = 1,\ F = \infty,\ G = \frac{1-\omega}{2},\ H = 1-\omega,\ I = \frac{2-\omega}{2},\ A' = 0,\ B' = \frac{2\omega-1}{7},\ C' = \frac{3\omega-2}{8},\ D' = \frac{3\omega-1}{8},\ E' = \frac{\omega}{2},\ F' = \frac{\omega-1}{2},\ G' = \infty$, and $I' = \omega$.

The necessary identifications are:

$$\begin{pmatrix} 1 & \omega \\ -\omega & 3-\omega \end{pmatrix} : \{AFHG\} \to \{C'F'E'D'\}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} : \{CDEF\} \to \{H'F'A'G'\},$$
$$\begin{pmatrix} \omega-1 & \omega+2 \\ \omega+2 & 5-4\omega \end{pmatrix} : \{G'H'I'\} \to \{D'B'C'\}, \begin{pmatrix} 1-2\omega & \omega+2 \\ -2-\omega & 3-\omega \end{pmatrix} : \{GHI\} \to \{BDC\},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \{ABCF\} \to \{A'E'I'G'\}, \begin{pmatrix} -\omega-1 & \omega \\ \omega-2 & 1 \end{pmatrix} : \{ABDE\} \to \{I'H'F'E'\},$$
$$\begin{pmatrix} -1 & 1 \\ 2\omega-2 & 1-2\omega \end{pmatrix} : \{AEIG\} \to \{B'A'E'D'\}, \begin{pmatrix} 1 & \omega-1 \\ -\omega & 3 \end{pmatrix} : \{FHIE\} \to \{F'A'B'C'\}.$$

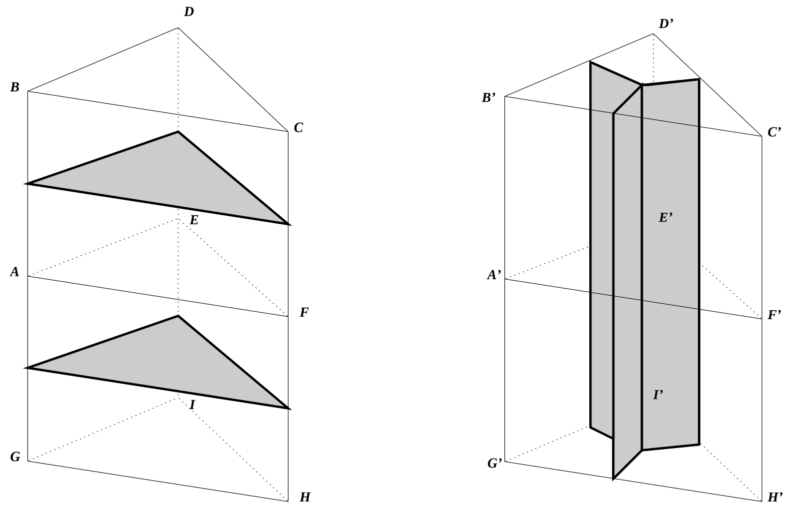

FIGURE 3. Subcomplex of the Mendoza complex supporting a non-zero infinitesimal deformation

As expected, $PH^1(\Gamma, \mathbb{R}^4_1)$ is again one-dimensional, but examining the kernel of the boundary operator shows that the classes are supported on the subcomplex of X shown (combinatorially) in figure 3. Note that this is a piecewise totally geodesic surface with one intersection curve and one complementary region. It appears to be distinct from the cycles one gets from any of the immersed totally geodesic surfaces in the link complement.

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