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LOCAL RIGIDITY OF HYPERBOLIC 3-MANIFOLDS AFTER DEHN SURGERY

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Abstract

It is well known that some lattices in $\mathrm{SO}(n, 1)$ *can be nontrivially deformed when included in* $\mathrm{SO}(n+1, 1)$ *(e.g., via bending on a totally geodesic hypersurface); this contrasts with the (super) rigidity of higher rank lattices. M. Kapovich recently gave the first examples of lattices in* $\mathrm{SO}(3, 1)$ *which are locally rigid in* $\mathrm{SO}(4, 1)$ *by considering closed hyperbolic* 3*-manifolds obtained by Dehn filling on hyperbolic two-bridge knots. We generalize this result to Dehn filling on a more general class of one-cusped finite volume hyperbolic* 3*-manifolds, allowing us to produce the first examples of closed hyperbolic* 3*-manifolds which contain embedded quasi-Fuchsian surfaces but are locally rigid in* $\mathrm{SO}(4, 1)$.

1. Introduction

This paper continues our study of the local deformation theory of rank-one lattices, which began with [28]. We are particularly interested in the local deformation space of representations of an $\mathrm{SO}(3, 1)$ lattice when viewed as a "Fuchsian" subgroup of $\mathrm{SO}(4, 1)$. The first examples of such deformations were given by B. Apanasov [2], [4] around the same time that W. Thurston introduced his closely related notion of bending deformations of Fuchsian groups (see [30, §8.7.3]). Examples in all dimensions and detailed discussion can be found in [13] (see also [22], [23]). Generalizations of bending have been considered by various authors (see, e.g., [3], [5], [29]) and typically involve either intersecting totally geodesic surfaces or a family of totally geodesic surfaces with a common boundary geodesic.

Kapovich conjectured in [16] that a closed hyperbolic 3-orbifold admits a nontrivial deformation in $O(4, 1)$ if and only if it contains an embedded quasi-Fuchsian suborbifold. In [28] we gave examples of infinitesimal deformations for infinitely many two-generator, closed hyperbolic 3-manifolds with zero first Betti number. One of these examples is non-Haken, and its fundamental group contains no nonelementary Fuchsian subgroups, providing an infinitesimal counterexample to one half of

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Kapovich's conjecture. One of the goals of this paper is to give counterexamples to the other half, by constructing infinitely many closed hyperbolic 3-manifolds that contain quasi-Fuchsian surfaces but are locally rigid in $\mathrm{SO}(4, 1)$. These manifolds also serve as counterexamples to a second conjecture of Kapovich [15], [17], namely, that for a closed 3-manifold, the deformation space of flat conformal structures should have only finitely many components. That this statement is false for hyperbolic n-manifolds, $n \geq 5$, was known previously (see [20]).

THEOREM 5.2
There exist infinitely many closed hyperbolic 3*-manifolds that contain embedded quasi-Fuchsian surfaces and that are locally rigid in* $\mathrm{SO}(4, 1)$*. The deformation space of flat conformal structures for these manifolds contains an infinite set of isolated points.*

Our method for producing these examples rests on the following general rigidity theorem.

THEOREM 4.4
Let $M \approx \Gamma\backslash\mathbb{H}^3$ be a complete, orientable, hyperbolic 3*-manifold of finite volume with one cusp. If $PH^1(\Gamma, \mathfrak{so}(4, 1)) = 0$, then there exist infinitely many closed hyperbolic* 3*-manifolds obtained by Dehn filling on M which are locally rigid in* $\mathrm{SO}(4, 1)$.

When M is the complement of a hyperbolic two-bridge knot in $\mathbb{S}^3$, this result was obtained by Kapovich in [18]. In this case, the vanishing of the parabolic cohomology $PH^1(\Gamma, \mathfrak{so}(4, 1))$ follows easily from the fact that Γ is generated by two parabolic elements (see §3). In general, this cohomology group can be computed from an ideal triangulation of M, and our calculations indicate that it vanishes quite often. We discuss some of these computations in §5.

2. Preliminaries

We first recall some of the notation from [28]. Let $\rho_0 : \pi \xrightarrow{\cong} \Gamma \subset \mathrm{SO}_0(3, 1)$ be the inclusion of a lattice in the identity component of $O(3, 1)$, and consider the composition of ρ_0 with the inclusion $\mathrm{SO}_0(3, 1) \hookrightarrow \mathrm{SO}_0(4, 1)$. We are interested in describing a neighborhood of ρ_0 in the representation space $\mathrm{Hom}(\pi, \mathrm{SO}_0(4, 1))$. This space is a real algebraic variety in a natural way, and the Zariski tangent space at ρ_0 is identified with the vector space of group cocycles $Z^1(\pi, \mathfrak{so}(4, 1))$. Here the coefficients lie in the Lie algebra of $\mathrm{SO}_0(4, 1)$, made into a $\mathbb{Z}\pi$-module via ρ_0 and the adjoint action. There is an action of $\mathrm{SO}_0(4, 1)$ on $\mathrm{Hom}(\pi, \mathrm{SO}_0(4, 1))$ by conjugation; the subspace $B^1(\pi, \mathfrak{so}(4, 1))$ of coboundaries consists of the Zariski tangent vectors along orbits of this action. For this reason, we call a nonzero cohomology class in $H^1(\pi, \mathfrak{so}(4,1))$ an *infinitesimal deformation of* ρ_0 *in* $\mathrm{SO}_0(4,1)$. An infinitesimal deformation is *integrable* if it is tangent to a nontrivial curve in $\mathrm{Hom}(\pi, \mathrm{SO}_0(4,1))$.

For any $\mathbb{Z}\pi$-module V, we compute the group cohomology $H^1(\pi, V)$ in terms of the standard resolution. Thus, a 1-cocycle is a function $c : \pi \to V$ satisfying $c(gh) = c(g) + gc(h)$ for all $g, h \in \pi$, and a 1-coboundary is a 1-cocycle of the form $c(g) = (1-g)w$ for some $w \in V$. When there is a possibility of confusion, we make the $\mathbb{Z}\pi$-module structure explicit; for example, we write V_ρ for a given $\rho : \pi \to \mathrm{Aut}(V)$.

When Γ is a nonuniform lattice, it contains parabolic elements, and we need a notion of cohomology classes that are trivial when restricted to cyclic parabolic subgroups. Let

$$PZ^1(\Gamma, V) = \left\{ c \in Z^1(\Gamma, V) \middle| \text{ for parabolic } \gamma \in \Gamma, c(\gamma) \in \mathrm{im}(1-\gamma) \right\},$$

and define the *parabolic cohomology*

$$PH^1(\Gamma, V) = PZ^1(\Gamma, V)/B^1(\Gamma, V).$$

In [7] it is shown that $PH^1(\Gamma, \mathfrak{so}(3,1)) = 0$ for a nonuniform lattice $\Gamma \subset \mathrm{SO}_0(3,1)$, giving the following analogue of the splitting lemma in [28].

LEMMA 2.1
Fix a representation $\rho_0 : \pi \to \mathrm{SO}_0(3,1) \hookrightarrow \mathrm{SO}_0(4,1)$. *The Lie algebra* $\mathfrak{so}(4,1)$ *splits as an* $\mathrm{SO}_0(3,1)$*-module* $\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathbb{R}^4_1$, *inducing a splitting in the parabolic cohomology*

$$PH^1\big(\pi, \mathfrak{so}(4,1)\big) \cong PH^1\big(\pi, \mathfrak{so}(3,1)\big) \oplus PH^1(\pi, \mathbb{R}^4_1).$$

When ρ_0 *is an isomorphism onto a nonuniform lattice in* $\mathrm{SO}_0(3,1)$, *we have*

$$PH^1\big(\pi, \mathfrak{so}(4,1)\big) \cong PH^1(\pi, \mathbb{R}^4_1).$$

The next results are standard consequences of duality for surfaces and 3-manifolds.

LEMMA 2.2 ([8])
Let π *be the fundamental group of a closed, orientable surface of genus* g. *Let* G *be a semisimple Lie group, and fix a representation* $\rho_0 : \pi \to G$. *Then*

$$\dim H^1(\pi, \mathfrak{g}) = (2g-2)\dim G + 2\dim H^0(\pi, \mathfrak{g}).$$

Proof
The Killing form on $\mathfrak{g}$ is nondegenerate and Ad-invariant, and therefore gives a G-module isomorphism between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. By Poincaré duality, we see that

$$H^2(\pi, \mathfrak{g}) \cong H^0(\pi, \mathfrak{g}^*)^* \cong H^0(\pi, \mathfrak{g})^*$$

and hence that

$$\dim H^2(\pi, \mathfrak{g}) = \dim H^0(\pi, \mathfrak{g}).$$

The Euler characteristic in group cohomology is independent of the representation ρ_0, so by considering the trivial representation, we see that it is equal to $(2-2g)\dim G$. The lemma follows. □

Let M be a compact, oriented 3-manifold with a nonempty collection of boundary components $\{\Sigma_j\}$, and let $\pi = \pi_1 M$. We write $H^*(\partial, V) = \bigoplus_j H^*(\pi_1 \Sigma_j, V)$ for the group cohomology of the boundary components, and we write $i^* : H^1(\pi, V) \to H^1(\partial, V)$ for the corresponding restriction map on first cohomology.

The next proposition is an easy consequence of Lefschetz duality for 3-manifolds and has appeared in [11, §15], [16], and [12]. A geometric argument for the case $G = \mathrm{SO}_0(3,1) \cong \mathrm{PSL}(2, \mathbb{C})$ can be found in [30, §5.6] and [6].

PROPOSITION 2.3
Let M be a compact, oriented 3-manifold such that ∂M consists of a nonempty union of tori. For any representation $\rho_0 : \pi \to G$ of $\pi = \pi_1 M$ in a semisimple Lie group G,

$$\dim H^1(\pi, \mathfrak{g}) = \dim \ker i^* + \dim H^0(\partial, \mathfrak{g}).$$

Proof (following [12])
For $j = 0, 1$ we have the well-known identifications $H^j(\pi, \mathfrak{g}) \cong H^j(M; E)$ and $H^j(\partial, \mathfrak{g}) \cong H^j(\partial M; E)$, where E is the flat bundle associated to the representation ρ_0. Consider the following commutative diagram, suppressing the local coefficients E:

$$\begin{array}{ccccc}
H^1(M) & \xrightarrow{i^*} & H^1(\partial M) & \xrightarrow{\delta^*} & H^2(M, \partial M) \\
\downarrow & & \downarrow & & \downarrow \\
H^2(M, \partial M)^* & \xrightarrow{\delta_*} & H^1(\partial M)^* & \xrightarrow{i_*} & H^1(M)^*
\end{array}$$

Here the vertical arrows are isomorphisms by duality, and the maps i_* and δ_* are the duals of i^* and δ^*, respectively. We have

$$\dim \operatorname{im} \delta^* = \dim \operatorname{im} i_* = \dim \operatorname{im} i^*,$$

and so

$$\begin{aligned}\dim H^1(\partial M) &= \dim\ker\delta^* + \dim\operatorname{im}\delta^* \\ &= 2\dim\operatorname{im} i^*.\end{aligned}$$

Combining this with Lemma 2.2, we obtain $\dim\operatorname{im} i^* = \dim H^0(\partial M)$ and thus

$$\begin{aligned}\dim H^1(M) &= \dim\ker i^* + \dim\operatorname{im} i^* \\ &= \dim\ker i^* + \dim H^0(\partial M).\end{aligned}$$ □

3. Subgroups generated by parabolics

The results in this section are not required for the proof of our main result, but help to explain the significance of the parabolic cohomology hypothesis and its relationship to the results of [18].

PROPOSITION 3.1

If Δ *is a subgroup of* $\mathrm{SO}_0(3,1)$ *generated by two parabolics, then* $PH^1(\Delta, \mathbb{R}^4_1) = 0$.

Proof

Let α_0 and β_0 be the parabolic generators in $\mathrm{SO}_0(3,1)$, and let $c \in PZ^1(\Delta, \mathbb{R}^4_1) \subseteq PZ^1(\Delta, \mathfrak{so}(4,1))$, using the splitting of Lemma 2.1. Now $c(\alpha_0) = (1-\alpha_0)v_0$ and $c(\beta_0) = (1-\beta_0)w_0$ for some $v_0, w_0 \in \mathfrak{so}(4,1)$, and we can define two curves of parabolics $\alpha_t = \exp(tv_0)\alpha_0\exp(tv_0)^{-1}$ and $\beta_t = \exp(tw_0)\beta_0\exp(tw_0)^{-1}$ in $\mathrm{SO}_0(4,1)$. But any pair of unipotents in $\mathrm{SO}_0(4,1)$ leaves invariant a round $\mathbb{S}^2$ in $\mathbb{S}^3$ (see [18]); hence the group generated by α_t and β_t is conjugate back into $\mathrm{SO}_0(3,1)$. This implies that $[c] \in PH^1(\Delta, \mathfrak{so}(3,1))$ and hence that $[c] = 0$. □

COROLLARY 3.2

If $M \approx \Gamma\backslash\mathbb{H}^3$ *is the complement of a hyperbolic two-bridge knot or link in* $\mathbb{S}^3$*, then* $PH^1(\Gamma, \mathfrak{so}(4,1)) = 0$.

Proof

It is well known that the fundamental group of a two-bridge knot or link is generated by two meridional loops, so in the hyperbolic case, Γ is generated by two parabolic elements. The corollary follows from Lemma 2.1 and the previous proposition. □

In fact, it is a remarkable consequence of the Smith conjecture that any finite volume, orientable, hyperbolic 3-manifold with fundamental group generated by two parabolic elements is the complement of a two-bridge link in $\mathbb{S}^3$ (see [1]).

COROLLARY 3.3

Let $M \approx \Gamma\backslash\mathbb{H}^3$ *be a complete, orientable, hyperbolic* 3*-manifold of finite volume with at least one cusp, and let* $i^* : H^1(\Gamma, \mathfrak{so}(4,1)) \to H^1(\partial, \mathfrak{so}(4,1))$ *be the restriction homomorphism as above. Then*

$$PH^1\big(\Gamma, \mathfrak{so}(4,1)\big) = \ker i^*.$$

Proof

That $\ker i^*$ is included in $PH^1(\Gamma, \mathfrak{so}(4,1))$ is clear from the definitions. For the opposite inclusion, we may assume that the coefficients of a representative cocycle c lie in $\mathbb{R}^4_1$ by Lemma 2.1. But $PH^1(\partial, \mathbb{R}^4_1) = 0$ by applying Proposition 3.1 to each boundary component, so $i^*[c] = 0$. □

4. Dehn surgery

We proceed with the proof of the rigidity theorem in this section. The basic strategy is as follows: for a finite volume complete hyperbolic 3-manifold with one cusp, the $\mathrm{PSL}(2,\mathbb{C})$ character variety $\mathfrak{R}$ is (real) 2-dimensional and smooth at the holonomy representation ρ_0. For the larger variety of representations into $\mathrm{SO}(4,1)$, we may use the results of §2, Lemma 4.1, and the parabolic cohomology hypothesis to show that the dimension of the Zariski tangent space at ρ_0 is 3. The key technical point in the proof is contained in Proposition 4.3, which shows the existence of nonintegrable tangent vectors in the tangent space at ρ_0. From this result, we argue that the set of points in $\mathfrak{R}$ corresponding to potentially nonrigid surgered manifolds lies in a proper subvariety. But it is known that the set of points yielding closed hyperbolic manifolds is a Zariski-dense subset of $\mathfrak{R}$, and the theorem follows. In the proof below, we actually work with the full representation variety (before conjugation) to avoid the difficulties involved in passing to the quotient variety.

The idea for the last part of the argument is taken from Kapovich's proof for the case of two-bridge knots (see [18]). We conjecture that the "infinitely many" in the conclusion of the rigidity theorem can be replaced by "all but finitely many."

LEMMA 4.1

Fix a representation $\rho_0 : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{SO}_0(3,1)$ *whose image is not generated by commuting order-two elliptics. Then* $\dim H^0(\mathbb{Z} \oplus \mathbb{Z}, \mathfrak{so}(3,1)) = 2$. *If, furthermore, the image of* ρ_0 *is not contained in any one-parameter subgroup, we have*

$$\dim H^0(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{R}^4_1) = \begin{cases} 1 & \textit{at a parabolic representation,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof

Having excluded the case of commuting order-two elliptics, it is well known that two nontrivial elements of $\mathrm{SO}_0(3,1)$ commute if and only if their fixed point sets in $\mathbb{S}^2 \approx \partial\mathbb{H}^3$ coincide. Thus, if one element is parabolic, they all are, and $\dim H^0(\mathbb{Z}\oplus\mathbb{Z}, \mathfrak{so}(3,1)) = 2$. Alternatively, there is an invariant geodesic line in $\mathbb{H}^3$, and once again we have $\dim H^0(\mathbb{Z}\oplus\mathbb{Z}, \mathfrak{so}(3,1)) = 2$.

In the case of a parabolic representation, the common parabolic fixed point corresponds to a fixed null vector in $\mathbb{R}^4_1$. On the other hand, there are clearly no fixed timelike vectors, and because the image is not contained in a one-parameter subgroup of parabolics, there is no invariant $\mathbb{H}^2 \subset \mathbb{H}^3$, hence no fixed spacelike vector. Thus, $\dim H^0(\mathbb{Z}\oplus\mathbb{Z}, \mathbb{R}^4_1) = 1$.

Otherwise, our assumption implies that $\rho_0(\mathbb{Z}\oplus\mathbb{Z})$ contains some loxodromic elements (throughout, we use *loxodromic* to include purely hyperbolic elements). It follows that there can be no fixed null or timelike vectors in $\mathbb{R}^4_1$ and that any invariant $\mathbb{H}^2 \subset \mathbb{H}^3$ contains the common invariant geodesic. But this is only possible if each element is purely hyperbolic, which forces $\rho_0(\mathbb{Z}\oplus\mathbb{Z})$ to lie within a one-parameter subgroup. We conclude that $\dim H^0(\mathbb{Z}\oplus\mathbb{Z}, \mathbb{R}^4_1) = 0$ in this case. □

LEMMA 4.2

Let α and β be generators of $\mathbb{Z}\oplus\mathbb{Z}$, and let $\rho \in \mathrm{Hom}(\mathbb{Z}\oplus\mathbb{Z}, \mathrm{SO}_0(4,1))$ be a representation such that $\rho(\alpha)$ is loxodromic and -1 is not an eigenvalue of $\rho(\beta)$. Then the image of ρ is conjugate into $\mathrm{SO}_0(3,1)$.

Proof

View $\mathrm{SO}_0(4,1)$ as acting by Möbius transformations on $\mathbb{R}^3 \cup \{\infty\}$ and conjugate so that $\mathbf{0}$ and ∞ are, respectively, the repelling and attracting fixed points of $\rho(\alpha)$. It follows that $\rho(\alpha)$ is of the form

$$x \mapsto \lambda_1 A x$$

for some $\lambda_1 > 1$ and $A \in \mathrm{SO}(3)$. Since $\rho(\beta)$ commutes with $\rho(\alpha)$, it leaves invariant the fixed set $\{\mathbf{0}, \infty\}$. In fact, $\rho(\beta)$ must fix $\mathbf{0}$ and ∞, for if it interchanged them, it would have an eigenvalue of -1. Thus, $\rho(\beta)$ is of the form

$$x \mapsto \lambda_2 B x,$$

where $\lambda_2 > 0$ and B commutes with A in $\mathrm{SO}(3)$. Since $\rho(\beta)$ does not have -1 as an eigenvalue, we can exclude the possibility that A and B are a pair of commuting half-turns, and so A and B share a common axis of rotation. We conclude that $\rho(\alpha)$ and $\rho(\beta)$ leave invariant a common plane in $\mathbb{R}^3$ and therefore that ρ is conjugate into $\mathrm{SO}_0(3,1)$. □

PROPOSITION 4.3

Let $\rho_0 : \pi \to \mathrm{SO}_0(3,1) \hookrightarrow \mathrm{SO}_0(4,1)$ be the inclusion of a torsion-free, nonuniform *lattice. Then the representation variety* $\mathrm{Hom}(\pi, \mathrm{SO}_0(4,1))$ *is singular at* ρ_0*. Indeed, given any* $v \in Z^1(\pi, \mathbb{R}^4_1)$ *such that* $i^*[v] \neq 0 \in H^1(\partial, \mathbb{R}^4_1)$*, there is a nonintegrable cocycle in* $Z^1(\pi, \mathfrak{so}(3,1)) \oplus \langle v \rangle$.

Proof
First, note that the existence of a cocycle v satisfying the properties in the statement of the proposition follows from Lemma 4.1 and Proposition 2.3. Since $i^*[v] \neq 0$, we may fix one cusp end T such that $i^*[v] \neq 0$ in $H^1(\pi_1 T, \mathbb{R}^4_1)$. Let α and β be generators for $\pi_1 T$.

A generic deformation $\omega_t \in \mathrm{Hom}(\pi, \mathrm{SO}_0(3,1))$ of $\omega_0 = \rho_0$ has the property that $\omega_t(\alpha)$ is loxodromic for small $t > 0$ (see [27]); fix one such deformation, and let $\dot{\omega} \in Z^1(\pi, \mathfrak{so}(3,1))$ be the tangent vector at $t = 0$. Since the loxodromic elements are open in $\mathrm{SO}_0(4,1)$, there exists $\epsilon > 0$ such that if ρ_t integrates $\dot{\omega} + \epsilon v$, then $\rho_t(\alpha)$ is loxodromic for small $t > 0$. Furthermore, since $\rho_0(\beta)$ is unipotent, $\rho_t(\beta)$ does not have -1 as an eigenvalue for small values of t. Thus, Lemma 4.2 implies that $\rho_t|_{\pi_1 T}$ is conjugate into $\mathrm{SO}_0(3,1)$ for small t. We conclude that $i^*(\dot{\omega} + \epsilon v)$, and therefore $i^*(v)$, are cohomologous to cocycles in $Z^1(\pi_1 T, \mathfrak{so}(3,1))$, a contradiction. □

We are now ready to prove the main theorem.

THEOREM 4.4
Let $M \approx \Gamma \backslash \mathbb{H}^3$ *be a complete, orientable, hyperbolic* 3*-manifold of finite volume with one cusp. If* $PH^1(\Gamma, \mathfrak{so}(4,1)) = 0$*, then there exist infinitely many closed hyperbolic* 3*-manifolds obtained by Dehn filling on* M *which are locally rigid in* $\mathrm{SO}(4,1)$.

Proof
By abuse of notation we also write M for the compact manifold with boundary whose interior is homeomorphic to $\Gamma \backslash \mathbb{H}^3$. As usual, we let $\pi = \pi_1 M$, and we fix a holonomy representation $\rho_0 : \pi \xrightarrow{\cong} \Gamma \subset \mathrm{SO}_0(3,1)$ corresponding to the complete hyperbolic structure on M. For brevity, we write $\mathfrak{X}_n = \mathrm{Hom}(\pi, \mathrm{SO}_0(n,1))$ for $n = 3, 4$. We begin with some properties of $\mathfrak{X}_3$.

First, recall that $\mathfrak{X}_3$ is smooth at ρ_0 (see [11, §15], [19, §8.8]); its dimension can be computed using Proposition 2.3:

$$\begin{aligned}
\dim Z^1\big(\pi, \mathfrak{so}(3,1)_{\rho_0}\big) &= \dim H^1\big(\pi, \mathfrak{so}(3,1)_{\rho_0}\big) + \dim B^1\big(\pi, \mathfrak{so}(3,1)_{\rho_0}\big) \\
&= \dim \ker i^* + \dim H^0\big(\partial, \mathfrak{so}(3,1)_{\rho_0}\big) + 6 \\
&= 0 + 2 + 6 = 8,
\end{aligned}$$

where $\ker i^*$ vanishes by [7], and we have used Lemma 4.1 for the $\mathbb{Z} \oplus \mathbb{Z}$ centralizer in $\mathrm{SO}_0(3,1)$. (Of course, since $\mathrm{SO}_0(3,1) \cong \mathrm{PSL}(2, \mathbb{C})$, it is more common to realize

$\mathfrak{X}_3$ as a smooth 4-dimensional complex variety, but as we are in the context of an inclusion into the noncomplex group $SO_0(4,1)$, it is convenient to work only with the real algebraic structure.) A neighborhood of ρ_0 in $\mathfrak{X}_3$ consists of representations with Zariski dense image in $SO_0(3,1)$; in particular, they all have trivial centralizer in $SO_0(4,1)$. It follows that the image $\widetilde{\mathfrak{X}}_3$ of the conjugation map $\mathfrak{X}_3 \times SO_0(4,1) \to \mathfrak{X}_4$ (i.e., the set of representations with image conjugate into $SO_0(3,1)$) is smooth and 12-dimensional near ρ_0.

On the other hand, the dimension of the Zariski tangent space to $\mathfrak{X}_4$ can be computed in a similar fashion:

$$\begin{aligned}\dim Z^1\big(\pi, \mathfrak{so}(4,1)_{\rho_0}\big) &= \dim H^1\big(\pi, \mathfrak{so}(4,1)_{\rho_0}\big) + \dim B^1\big(\pi, \mathfrak{so}(4,1)_{\rho_0}\big)\\ &= \dim\ker i^* + \dim H^0\big(\partial, \mathfrak{so}(4,1)_{\rho_0}\big) + 10\\ &= 0 + 3 + 10 = 13,\end{aligned}$$

where we have used the parabolic cohomology hypothesis for the vanishing of $\ker i^*$ and Lemma 4.1 for the $\mathbb{Z}\oplus\mathbb{Z}$ centralizer in $SO_0(4,1)$.

Because $\mathfrak{X}_4$ is singular at ρ_0 (see Proposition 4.3), the dimension of $\mathfrak{X}_4$ as a real algebraic variety is strictly less than 13. Since $\widetilde{\mathfrak{X}}_3$ is smooth and 12-dimensional, we may conclude that the dimension of $\mathfrak{X}_4$ is precisely 12. Thus, at a representation $\rho \in \mathfrak{X}_4$ near ρ_0, we have $\dim Z^1(\pi, \mathfrak{so}(4,1)_\rho) = 12 + \varepsilon(\rho)$, where $\varepsilon(\rho) = 0, 1$. A calculation like the one above shows that $\dim H^1(\pi, \mathfrak{so}(4,1)_\rho) = 2 + \varepsilon(\rho)$ and therefore that $\dim H^1(\pi, (\mathbb{R}^4_1)_\rho) = \varepsilon(\rho)$ for $\rho \in \mathfrak{X}_3$.

To dispose of the possibility that there is an open neighborhood of ρ_0 where $\varepsilon(\rho) = 1$, we replace $\mathfrak{X}_4$ with the reduced variety $\mathfrak{Y}_4$ defined by the ideal of polynomials vanishing on $\mathfrak{X}_4$. These two varieties coincide as point sets, but the Zariski tangent spaces of $\mathfrak{Y}_4$ are a priori smaller, and we are able to conclude that its singular subvariety $\mathfrak{B}$ has positive codimension (see [32]).

We next write $\mathfrak{R}$ for the 2-dimensional character variety of representations in $SO_0(3,1)$ up to conjugation. Fix a basis for the homology of the boundary torus ∂M, and write $\boldsymbol{\kappa}(\rho) = (p,q)$ for the generalized Dehn surgery invariant associated to a representation ρ as in [27]. The set of conjugacy classes of $\rho \in \mathfrak{R}$ such that $\boldsymbol{\kappa}(\rho)$ is a pair of relatively prime integers clusters at ρ_0 and is Zariski-dense (see [18], [27]); the same statement holds in $\widetilde{\mathfrak{X}}_3$. Since $\mathfrak{B}$ has positive codimension, we conclude that there are infinitely many closed hyperbolic 3-manifolds obtained by (p,q) filling on M such that the corresponding representation ρ is not contained in $\mathfrak{B}$.

Fix one such representation $\rho \notin \mathfrak{B}$ with $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\rho) = (p,q)$, and let $M_{\boldsymbol{\kappa}}$ be the closed manifold obtained by (p,q) filling on M. The representation ρ factors through $\pi_1 M_{\boldsymbol{\kappa}}$ to give the holonomy $\overline{\rho}$ of the complete hyperbolic structure on $M_{\boldsymbol{\kappa}}$, so we have $Z^1(\pi_1 M_{\boldsymbol{\kappa}}, (\mathbb{R}^4_1)_{\overline{\rho}}) \subseteq Z^1(\pi, (\mathbb{R}^4_1)_\rho)$. The image $\rho(\pi) = \overline{\rho}(\pi_1 M_{\boldsymbol{\kappa}})$ is a lattice in $SO_0(3,1)$ and therefore has trivial centralizer in $SO_0(4,1)$. This means that

$B^1(\pi_1 M_\kappa, (\mathbb{R}^4_1)_{\overline{\rho}}) = B^1(\pi, (\mathbb{R}^4_1)_\rho) \cong \mathbb{R}^4_1$, and so

$$H^1\big(\pi_1 M_\kappa, (\mathbb{R}^4_1)_{\overline{\rho}}\big) \subseteq H^1\big(\pi, (\mathbb{R}^4_1)_\rho\big).$$

Since $H^1(\pi_1 M_\kappa, \mathfrak{so}(4,1)_{\overline{\rho}}) = H^1(\pi_1 M_\kappa, (\mathbb{R}^4_1)_{\overline{\rho}})$ by the splitting lemma in [28], we see by [31] that $\overline{\rho}$ is locally rigid whenever $\varepsilon(\rho) = 0$. On the other hand, if there exists a nontrivial integrable deformation of $\overline{\rho}$ with tangent vector $v \in H^1(\pi_1 M_\kappa, (\mathbb{R}^4_1)_{\overline{\rho}})$, it would also define a nontrivial curve in $\mathfrak{Y}_4$ starting at ρ. We conclude from this that the Zariski tangent space to $\mathfrak{Y}_4$ at ρ is at least 13-dimensional and therefore that $\rho \in \mathfrak{B}$, a contradiction. □

5. Examples and remarks

The hypothesis of vanishing parabolic cohomology in the main theorem is closely related to the Menasco-Reid conjecture (see [25]), which states that no hyperbolic knot complement in $\mathbb{S}^3$ contains a closed, embedded, totally geodesic surface. Of course, the existence of such a surface implies the existence of a nontrivial class in $PH^1(\Gamma, \mathfrak{so}(4,1))$ by bending. The converse is false, however, as we have shown that the family of Turk's head links (starting with 8_{18}; see [28]) all have nonvanishing parabolic cohomology, while it is observed in [25] that no closed 3-braid can contain a closed embedded totally geodesic surface. The Fibonacci manifolds discussed in [28] are the two-fold branched covers of the Turk's head links, and the respective cohomology calculations are closely related.

We should also remark that computer-assisted calculations are possible using the Fox calculus and group representations computed from ideal triangulations with SnapPea. As an example, among knots in $\mathbb{S}^3$ with fewer than eleven crossings, we have found only three that have nontrivial parabolic cohomology. (Two of these are the Turk's head links 8_{18} and 10_{123}, and the third is 10_{99} in the standard tables.) Thus, vanishing results of this kind appear to be a promising approach to the Menasco-Reid conjecture; in addition, they can be used to produce many interesting closed examples in light of our main theorem.

Indeed, one of our main goals was to find counterexamples to the conjectures of Kapovich mentioned in the introduction. We may now do so by considering closed manifolds obtained by Dehn filling on certain hyperbolic knots in $\mathbb{S}^3$. For instance, the 3-braids $e_2^{-1}e_1^2e_2^{-3}e_1e_2^{-1}e_1^2$ and $e_2^{-1}e_1e_2^{-3}e_1e_2^{-2}e_1^2$ close up to the knots 10_{91} and 10_{94}, respectively, each of which is hyperbolic and satisfies $PH^1(\pi, \mathfrak{so}(4,1)) = 0$ using SnapPea.

PROPOSITION 5.1
All but finitely many Dehn fillings on 10_{91} and 10_{94} yield closed hyperbolic 3-manifolds that contain at least one closed, embedded, quasi-Fuchsian surface.

Proof

Let $K \subset \mathbb{S}^3$ be one of these knots, and let $M = \mathbb{S}^3 \setminus K$. First, Thurston's Dehn surgery theorem shows that all but finitely many Dehn fillings on K are hyperbolic. Using [24, Cor. 3.7], M contains a closed, orientable, incompressible surface Σ that remains incompressible after any nontrivial Dehn filling. We claim that when the resulting closed manifold M' is hyperbolic, the resulting incompressible surface $\Sigma' \subset M'$ is quasi-Fuchsian. If it were not, results of F. Bonahon and Thurston imply that Σ' would lift to a fiber in a fibration over $\mathbb{S}^1$ in some finite cover of M'. But then Σ' is either itself a fiber in a fibration of M' or separates M' into two twisted I-bundles over a nonorientable surface (see [26]). The first possibility can be excluded because Σ and Σ' are separating. To exclude the second possibility, observe that Σ separates $\mathbb{S}^3$ into two connected components, M_0 (containing the knot) and M_1 (not containing the knot). The manifold M_1 is not a twisted I-bundle over a nonorientable surface since $H^2(M_1, \mathbb{Z}) \cong \tilde{H}_0(M_0, \mathbb{Z}) = 0$; thus, the same is true in any manifold obtained by Dehn filling on K. □

THEOREM 5.2

There exist infinitely many closed hyperbolic 3*-manifolds that contain embedded quasi-Fuchsian surfaces and that are locally rigid in* $\mathrm{SO}(4,1)$. *The deformation space of flat conformal structures for these manifolds contains an infinite set of isolated points.*

Proof

By the previous proposition, infinitely many Dehn fillings on 10_{91} or 10_{94} are closed hyperbolic manifolds containing quasi-Fuchsian surfaces, and these are locally rigid in $\mathrm{SO}(4,1)$ by our main theorem.

For the second claim, we must use Thurston's holonomy theorem (see [10]), which states that the holonomy map

$$\mathrm{hol} : \mathscr{S}(M) \to \mathrm{Hom}\big(\Gamma, \mathrm{SO}(4,1)\big) / \mathrm{SO}(4,1)$$

from the deformation space of flat conformal structures on M to the representation variety is an open map and lifts to a local homeomorphism from the space of Möbius developing maps to $\mathrm{Hom}(\Gamma, \mathrm{SO}(4,1))$. Fix a flat conformal structure $\sigma \in \mathscr{S}(M)$ with $\rho = \mathrm{hol}(\sigma)$ Fuchsian (the inclusion of an $\mathrm{SO}(3,1)$ lattice). Since ρ is a stable representation (see [13, §1]), there exist neighborhoods U of σ and V of ρ, and open sets $\tilde{U}$ and $\tilde{V}$ such that U (resp., V) is the quotient of $\tilde{U}$ (resp., $\tilde{V}$) by the (finite) isotropy of σ (resp., ρ), and hol lifts to a homeomorphism from $\tilde{U}$ to $\tilde{V}$. In particular, if ρ is isolated, it follows that σ is isolated as well. In our setup, hol is actually two-to-one: the isotropy of ρ in $\mathrm{SO}(4,1)$ has order two, generated by the inclusion of

$-I \in SO(3,1)$ into $SO(4,1)$, while the isotropy of σ is trivial (using the main result of [14] to see that σ is not fixed by the inclusion of $-I$).

In [9], W. Goldman gave a construction that allows one to perform "$2\pi n$-grafting" on a quasi-Fuchsian surface in a hyperbolic 3-manifold, yielding an infinite family of distinct flat conformal structures with the same (Fuchsian) holonomy representation. When the Fuchsian representation is locally rigid in $SO(4,1)$, as in the examples constructed above, the flat conformal structures produced by Goldman's construction are isolated in $\mathscr{S}(M)$. $\square$

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