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A one-dimensional embedding complex

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Abstract

We give the first explicit computations of rational homotopy groups of spaces of "long knots" in Euclidean spaces. We define a spectral sequence which converges to these rational homotopy groups whose E^1 term is defined in terms of familiar Lie algebras. For odd k we establish a vanishing line for this spectral sequence, show the Euler characteristic of the rows of this E^1 term is zero, and make calculations of E^2 in a finite range. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we introduce a spectral sequence which converges to the rational homotopy groups of $Emb(I, \mathbb{R}^k \times I)$, for $k \geqslant 3$, which is the space of embeddings of an interval in $\mathbb{R}^k \times I$ with fixed endpoints and tangent vectors at those endpoints (essentially, the space of long knots in $\mathbb{R}^{k+1}$). Our starting point is the work of [11], which defines such spectral sequences in terms of the topology of configuration spaces. The paper [11] in turn builds upon work of Goodwillie and his collaborators [4,5,14], who have built a powerful theory for studying spaces of embeddings in general.

The rational homotopy groups of configuration spaces, which comprise the E^1 term, are Lie algebras which are well-known. Just as the study of cohomology of embedding

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spaces gives rise to the study of graph cohomology, which has been studied extensively [2,7,12,13], our complexes of Lie algebras are interesting new objects in quantum algebra. Similar complexes were described by Kontsevich in his plenary talk [8].

We start by reviewing the computation of the rational homotopy groups of ordered configurations of points in Euclidean space as a graded Lie algebra under Whitehead product, as well as some basics of free Lie algebras, which appear as subalgebras of these homotopy groups. At that point, we will have the necessary algebraic background to define the chain complexes which are the rows of the E^1 term of our spectral sequence. It turns out that through E^2, our spectral sequence for the homotopy groups of $Emb(I, \mathbb{R}^k \times I)$ depends, up to regrading, only on the parity of k. We focus on odd k. We prove some fundamental facts about these complexes, such as the vanishing of their Euler characteristic. We proceed to describe algorithms for computing the homology of these chain complexes, and in the final section present the results of these computations in low dimensions. In some cases, the classes which arise in E^2 must survive, implying the existence of non-trivial spherical families of embeddings. Non-zero higher differentials are also possible. We end with a brief discussion of the case of k even, which pertains to the theory of finite-type knot invariants.

2. The rational homotopy groups of configuration spaces

We remind the reader of the computations of the rational homotopy groups of configuration spaces [3] and their Lie algebra structure under Whitehead product. Throughout this paper, $\pi_*(X)$ will denote the homotopy groups of X tensored with the rational numbers. Let $F(M,n)$ denote the space of ordered configurations of n distinct points in a manifold M. We consider the projection $\rho : F(M,n) \to F(M,n-1)$ defined by forgetting the last point in the configuration, which is in fact a fiber bundle whose fiber is $M \setminus \{(n-1) \text{ points}\}$. Let ι denote the inclusion of the fiber. When $M = \mathbb{R}^{k+1}$, the fibers are homotopy equivalent to $\bigvee_{n-1} S^k$, and the projection map admits a section, by adding a point (say in a fixed direction at a large distance) to a configuration of $n-1$ points. This section leads to a splitting of the long exact sequence of a fibration into split short exact sequences

$$0 \to \pi_i\left(\bigvee_{n-1} S^k\right) \to \pi_i(F(\mathbb{R}^k \times I, n)) \to \pi_i(F(\mathbb{R}^{k+1}, n-1)) \to 0.$$

By induction, we find that additively

$$\pi_i(F(\mathbb{R}^k \times I, n)) \cong \bigoplus_{j=1}^{n-1} \pi_i\left(\bigvee_{j} S^k\right).$$

We now compute the structure of the rational homotopy groups $\pi_*(F(\mathbb{R}^k \times I, n))$ as a Lie algebra under the Whitehead product.

Definition 2.1. Let $\mathscr{B}_n^{\mathrm{o}}$ (resp. $\mathscr{B}_n^{\mathrm{e}}$) be the Lie algebra (super-Lie algebra for $\mathscr{B}_n^{\mathrm{e}}$) generated over $\mathbb{Q}$ by classes x_{ij} for $1 \leqslant i,j \leqslant n$ with relations

1. $x_{ij} = x_{ji}$ (resp. $-x_{ji}$ for $\mathscr{B}_n^{\mathrm{e}}$).
2. $x_{ii} = 0$.
3. $[x_{ij}, x_{\ell m}] = 0$ if $\{i,j\} \cap \{\ell, m\} = \emptyset$.
4. $[x_{ij}, x_{j\ell}] = [x_{j\ell}, x_{\ell i}] = [x_{\ell i}, x_{ij}]$.

We call $\mathscr{B}_n^{\mathrm{o}}$ and $\mathscr{B}_n^{\mathrm{e}}$ *configuration space Lie algebras.*

Theorem 2.1. *There is a Lie algebra isomorphism between $\pi_*(F(\mathbb{R}^k \times I, n))$ and $\mathscr{B}_n^{\mathrm{o}}$ if k is odd or $\mathscr{B}_n^{\mathrm{e}}$ if k is even.*

Proof. We first define classes which generate $\pi_*(F(\mathbb{R}^k \times I, n))$ as a Lie algebra. Pick a basepoint in $F(\mathbb{R}^k \times I, n)$, say with $z_i = (2i, 0, \ldots, 0)$ for definiteness. There are $\binom{n}{2}$ generators of $\pi_k(F(\mathbb{R}^k \times I, n))$, corresponding to distinct pairs $\{i,j\} \subseteq \{1, \ldots, n\}$, which we now realize geometrically. We define $b_{ij} \in \pi_k(F(\mathbb{R}^k \times I, n))$ as the class represented by the composite of two maps. First, we collapse S^k onto $S^k \vee I$ by sending the "southern hemisphere" of S^k to I through the height function. Next, choose a path γ_{ij} from z_i to the point $(2j-1, 0, \ldots, 0)$ in the complement of the other configuration points, and let ι_j denote the map which sends S^k to the unit sphere about the point z_j. To define b_{ij} we compose the collapse map above with the map $S^k \vee I$ to $F(\mathbb{R}^k \times I, n)$ which sends $t \in I$ to $F(\mathbb{R}^k \times I, n)$ as $(z_1, \ldots, z_{i-1}, \gamma_{ij}(t), z_{i+1}, \ldots, z_n)$ and S^k to $F(\mathbb{R}^k \times I, n)$ as $v \mapsto (z_1, \ldots, z_{i-1}, \iota_j(v), z_{i+1}, \ldots, z_n)$.

To see inductively that these classes are generators of $\pi_k(F(\mathbb{R}^k \times I, n))$, we simply note that b_{in} is equal to the image under ι_* of the generator of $\pi_k(\bigvee_{n-1} S^k)$ defined by the inclusion of the ith wedge factor.

It is simple to check that these b_{ij} satisfy the relations for x_{ij} in the definition of $\mathscr{B}_n^{\mathrm{o}}$. Note that from the usual graded commutativity of the Whitehead product, brackets in b_{ij} anti-commute when k is odd and commute when k is even. Note that $b_{ij} = (-1)^{k+1} b_{ji}$ and $b_{ii} = 0$ so that relations (1) and (2) are satisfied.

We next verify that the b_{ij} satisfy relation (3). Recall that if $\{f\}$ and $\{g\}$ are elements of $\pi_k(X)$ then $[\{f\}, \{g\}] = 0$ if and only if $f \vee g : S^k \vee S^k \to X$ extends to $S^k \times S^k$. If $\{i,j\} \cap \{\ell, m\} = \emptyset$, the map $b_{ij} \vee b_{\ell m}$ may be so extended by sending

$$v \times w \mapsto (z_1, \ldots, z_{i-1}, \iota_{ij}(v), z_{i+1}, \ldots, z_{\ell-1}, \iota_{\ell m}(w), z_{\ell+1}, \ldots),$$

where ι_{ij} is the composite of the collapse map of S^k onto $S^k \vee I$ with $\iota_j \vee \gamma_{ij}$. Informally we say that z_i can travel around z_j and z_ℓ can travel around z_m without having their paths (the images of S^k under the projection onto the ith and ℓth coordinates) intersect.

Next, we verify that the b_{ij} satisfy relation (4). Equivalently, we claim that $[b_{j\ell}, b_{ij} + b_{i\ell}] = 0$. Informally, we say that $b_{ij} + b_{i\ell}$ is represented by a map in which z_i travels around z_j and z_ℓ but no other points in the configuration, and this may happen simultaneously as z_j travels around z_ℓ, giving an extension of $(b_{ij} + b_{i\ell}) \vee b_{j\ell}$ to $S^k \times S^k$ similar to the one given for $b_{ij} \vee b_{\ell m}$.

We claim that relations (1)–(4) are a complete set of relations for $\pi_*(F(\mathbb{R}^k \times I, n))$. This follows from the fact that these relations may be used to reduce to an additive basis of Lie algebra monomials of the form $[\cdots[b_{im}, b_{jm}]\cdots b_{\ell m}\cdots]$, where $i, j, \ell < m \leqslant n$. We exhibit this claim algorithmically when discussing the computations in Section 5; see in particular Algorithm 5.2. □

The fiber sequence above leads us to identify some subalgebras of $\pi_*(F(\mathbb{R}^k \times I, n))$ which are free Lie algebras. Tensored with the rationals, the homotopy groups of wedges of spheres, $\pi_{*+1}(\bigvee_j S^k)$ for $k > 1$, are well known [6,15] to form free Lie algebra under Whitehead product, with j generators in degree $*+1 = k$. Since the inclusion map $\iota : \bigvee_{n-1} S^k \to F(\mathbb{R}^k \times I, n)$ is injective on homotopy, and by naturality of Whitehead products, the image of these homotopy groups under ι_* in $\pi_*(F(\mathbb{R}^k \times I, n))$ is a free Lie algebra which is generated by the classes b_{in}.

In the development of the spectral sequence we will in fact need the rational homotopy groups of $FT(k,n) = F(\mathbb{R}^k \times I, n) \times (S^k)^n$. We call $FT(k,n)$ the space of tangential configurations, thinking of the points in S^k as unit tangent vectors at points of a configuration. Recall that the homotopy groups of a product of spaces is a direct sum of their homotopy groups, and all Whitehead products between these summands are zero. Let Λ^{o} (resp. Λ^{e}) be the free Lie algebra (resp. super-Lie algebra) on one generator. Let $\mathscr{BT}_n^{\mathrm{o}}$ denote $\mathscr{B}_n^{\mathrm{o}} \oplus n\Lambda^{\mathrm{o}}$ and similarly for $\mathscr{BT}_n^{\mathrm{e}}$.

Corollary 2.2. *There is a Lie algebra isomorphism between $\pi_{*+1}(FT(k,n))$ and $\mathscr{BT}_n^{\mathrm{o}}$ if k is odd or $\mathscr{BT}_n^{\mathrm{e}}$ if k is even.*

These isomorphisms respect the gradings involved. We may grade $\mathscr{BT}_n^{\mathrm{o}}$ according to the number of generators appearing in a bracket. The dth graded summand of $\mathscr{BT}_n^{\mathrm{o}}$ coincides with $\pi_{d(k-1)+1}(FT(k,n))$.

3. Free Lie algebras

Let $\mathscr{L}(A)$ denote the free Lie algebra over $\mathbb{Q}$ on a set A of symbols. For our explicit computations, we must choose an additive basis for $\mathscr{L}(A)$. Natural labels for elements of free Lie algebras can be obtained from rooted, planar binary trees (hereafter, referred to as simply a *trees*) with leaves labeled by elements of A. Such a tree prescribes a bracketing of the elements which label the leaves. The number of leaves is the *degree* of the tree. Trees with a root but no branches (degree one) are identified with the set of symbols A. When the context is clear, we will identify trees with the free Lie algebra elements they produce. The obvious product of two trees x and y (a tree with a new root, left subtree x, and right subtree y) corresponds to the product in the Lie algebra and will therefore also be denoted $[x, y]$.

A set $\mathscr{H}$ of trees is called a *Hall set* for $\mathscr{L}(A)$ [9, Section 4.1] if the following conditions hold:

1. $\mathscr{H}$ has a total order $\leqslant$.
2. $A \subset \mathscr{H}$.
3. If $h = [h_1, h_2] \in \mathscr{H}$ then $h_2 \in \mathscr{H}$ and $h < h_2$.
4. For any tree $h = [h_1, h_2]$ of degree at least two, we have $h \in \mathscr{H}$ if and only if $h_1, h_2 \in \mathscr{H}$, $h_1 < h_2$, and either $h_1 \in A$ or $h_1 = [x, y]$ with $h_2 \leqslant y$.

It is straightforward to show that a Hall set forms an additive basis of $\mathscr{L}(A)$ [9, Theorem 4.9] (cf. Algorithm 5.1 below). The basis elements comprising a fixed Hall set will be called *Hall trees*. It is easy to see that (many) Hall sets exist [9, Proposition 4.1]; for completeness, we give a quick description of an algorithm for creating one.

Algorithm 3.1 (Generating a Hall set)**.** Given an ordered list of symbols A and a positive integer d, this algorithm outputs a list $\mathscr{H}_d$ consisting of the elements of degree less than or equal to d in a Hall set for $\mathscr{L}(A)$. The list $\mathscr{H}_d$ will be sorted according to the total order on the Hall set.

1. Set a counter $n = 1$. Copy the list A into $\mathscr{H}_d$.
2. If $n = d$, terminate the algorithm and output $\mathscr{H}_d$. Otherwise, proceed.
3. Form all products $[h_1, h_2]$ such that $h_1, h_2 \in \mathscr{H}_d$, $h_1 < h_2$, the degree of $[h_1, h_2]$ is $n+1$, and condition (4) is satisfied in the definition of Hall set. The only choice to be made is where to insert this new element in the ordering on $\mathscr{H}_d$; for definiteness in performing the calculations in Section 6, we insert $[h_1, h_2]$ into $\mathscr{H}_d$ as the immediate successor to h_1, thus $h_1 < [h_1, h_2] < h_2$ as required.
4. Increment n and go to (2).

Example. The output of Algorithm 3.1 with $A = \{a, b\}$ and $d = 5$ is the following list:

$$a \;\; [a,[[[a,b],b],b]] \quad [a,[a,[a,[a,b]]]] \quad [a,[a,[[a,b],b]]] \quad [a,[[a,b],b]],$$

$$[a,[a,[a,b]]] \quad [a,[a,b]] \quad [[a,[a,b]],[a,b]] \quad [a,b],$$

$$[[a,b],[[a,b],b]] \quad [[a,b],b] \quad [[[a,b],b],b] \quad [[[[a,b],b],b],b] \;\; b.$$

The following result will be used in computing the Euler characteristic of the chain complexes which appear as the rows of the E^1 term of our spectral sequence.

Lemma 3.1 ([9, Corollary 4.14])**.** *The number of Hall trees for $\mathscr{L}(A)$ of degree d equals*

$$\frac{1}{d}\sum_{j|d} \mu(j)|A|^{d/j},$$

where μ is the Möbius function.

4. The spectral sequence

In this section, we present an explicit realization of the spectral sequence introduced in [11, Section 4] which converges to the rational homotopy groups of $Emb(I, \mathbb{R}^k \times I)$. The spectral sequence arises from models of $Emb(I, \mathbb{R}^k \times I)$ which are reminiscent of cosimplicial spaces, but whose combinatorics are based on Stasheff polytopes instead of simplices. The entries are (Fulton–MacPherson compactified versions of) the ordered tangential configuration spaces $FT(k,n) = F(\mathbb{R}^k \times I, n) \times (S^k)^n$.

We can now describe the E^1 term of this spectral sequence. We first describe an unreduced version (which will be denoted throughout by the addition of a tilde $\tilde{E}^1$), followed by the reduced version (denoted simply E^1). Recall that one way to obtain the E^1 term of the homotopy spectral sequence of a cosimplicial space is by first passing to homotopy groups of the entries, which if all entries are simply connected defines a cosimplicial abelian group. The E^1 term is then the chain complex associated to this cosimplicial abelian group, which is bi-graded because the homotopy groups themselves are graded. We show in [11] that even though our models are based on Stasheff polyhedra, applying homotopy groups to these models gives rise to cosimplicial abelian groups. Hence the $\tilde{E}^1$ term of our spectral sequence is the chain complex of the cosimplicial abelian group

$$\pi_*(FT(k,0)) = pt. \begin{smallmatrix}\rightrightarrows\\ \leftarrow\end{smallmatrix} \pi_*(FT(k,1)) \begin{smallmatrix}\rightrightarrows\\ \leftleftarrows\end{smallmatrix} \pi_*(FT(k,2)) \cdots$$

Here the coface maps d^i_* are induced by maps d^i on configuration spaces (or rather their Fulton–MacPherson compactifications) which are "doubling" the ith point in a tangential configuration in the direction of the unit tangent vector determined by the ith factor of S^k, or if $i = 0$ or n by adding a point to the configuration at $(\vec{0}, 0)$ or $(\vec{0}, 1) \in \mathbb{R}^k \times I$. The codegeneracy maps s^i are defined by forgetting a point in the configuration.

Theorem 4.1 (see Sinha [11]). *There is a second-quadrant spectral sequence whose E^1 term is given by $\tilde{E}^1_{-p,q} = \pi_q(FT(k,p))$ and d^1 given by $\sum_i (-1)^i d^i_*$ which converges to $\pi_*(Emb(I, \mathbb{R}^k \times I))$.*

We now make the coface and codegeneracy maps algebraically explicit. Recall from Corollary 2.2 that the rational homotopy groups of $FT(k,n)$ are isomorphic to the Lie algebra $\mathscr{BT}^{\mathrm{o}}_n$ (or $\mathscr{BT}^{\mathrm{e}}_n$) generated by classes x_{ij} and y_i for $1 \leqslant i,j \leqslant n$ satisfying $x_{ii} = 0$, $x_{ij} = (-1)^{k+1} x_{ji}$, the other relations of Definition 2.1, and so that $[y_i, x_{j\ell}] = 0$ for all i,j,ℓ and $[y_i, y_j] = 0$ for $i \neq j$.

Definition 4.1. Define $\sigma^\ell(i)$ to be i if $i < \ell$ and $i+1$ if $i > \ell$. For $0 \leqslant \ell \leqslant n+1$ define $\partial^\ell : \mathscr{BT}^{\mathrm{o}}_n \to \mathscr{BT}^{\mathrm{o}}_{n+1}$ (respectively from $\mathscr{BT}^{\mathrm{e}}_n$ to $\mathscr{BT}^{\mathrm{e}}_{n+1}$) to be the Lie algebra homomorphism defined on generators as follows.

$$\partial^\ell(x_{ij}) = \begin{cases} x_{\sigma^\ell(i)\sigma^\ell(j)} & \text{if } i,j \neq \ell, \\ x_{i\sigma^\ell(j)} + x_{i+1,\sigma^\ell(j)} & \text{if } i = \ell, \end{cases}$$

$$\partial^\ell(y_i) = \begin{cases} y_{\sigma^\ell(i)} & \text{if } i \neq \ell, \\ x_{i,i+1} + y_i + y_{i+1} & \text{if } i = \ell. \end{cases}$$

Definition 4.2. For $1 \leqslant \ell \leqslant n$ define $\phi^\ell : \mathscr{BT}_n^{\mathrm{o}} \to \mathscr{BT}_{n-1}^{\mathrm{o}}$ (respectively from $\mathscr{BT}_n^{\mathrm{e}}$ to $\mathscr{BT}_{n-1}^{\mathrm{e}}$) to be the Lie algebra homomorphism defined on generators as follows.

$$\phi^\ell(x_{ij}) = \begin{cases} x_{\sigma^\ell(i)\sigma^\ell(j)} & \text{if } i, j \neq \ell, \\ 0 & \text{if } i \text{ or } j = \ell, \end{cases}$$

$$\phi^\ell(y_i) = \begin{cases} y_{\sigma^\ell(i)} & \text{if } i \neq \ell, \\ 0 & \text{if } i = \ell. \end{cases}$$

The following proposition is immediate from the definitions of the classes b_{ij} and the maps d^ℓ and s^ℓ.

Proposition 4.2. *Under the isomorphisms of Corollary* 2.2 *the homomorphisms d_*^ℓ and s_*^ℓ coincide with ∂^ℓ and ϕ^ℓ, respectively.*

Making Theorem 4.1 algebraically explicit using Corollary 2.2 and the previous proposition leads us to the following spectral sequence whose E^1 term is defined in terms of configuration space Lie algebras.

Corollary 4.3. *There is a second-quadrant spectral sequence which converges to $\pi_*(Emb(I, \mathbb{R}^k \times I))$ such that $\tilde{E}^1_{-n,d(k-1)+1}$ is isomorphic to the dth graded summand of $\mathscr{BT}_n^{\mathrm{o}}$ (resp. $\mathscr{BT}_n^{\mathrm{e}}$), $\tilde{E}^1_{-n,q} = 0$ when $q-1$ is not a multiple of $k-1$, and d^1 is given by $\sum_i (-1)^i \partial^i$.*

For the rest of the paper, we focus on the case in which k is odd.

Now we will describe a reduced version of the preceding spectral sequence. The standard reduction of a cosimplicial abelian group proceeds by replacing the nth group by the intersection of the kernels of the codegeneracy maps. Such a reduction does not change the homology of the associated chain complex. First note that the codegeneracy maps $\phi^\ell : \mathscr{BT}_n^{\mathrm{o}} \to \mathscr{BT}_{n-1}^{\mathrm{o}}$ respect the direct sum decomposition $\mathscr{BT}_n^{\mathrm{o}} = \mathscr{B}_n^{\mathrm{o}} \oplus n\Lambda^{\mathrm{o}}$. Restricted to the Λ^{o} factors, the intersection of the kernel of the ϕ^ℓ is zero unless n is equal to one, in which case it is all of Λ^{o}. Restricted to the $\mathscr{B}_n^{\mathrm{o}}$ factor, the kernel of the codegeneracy map $\phi^n : \mathscr{B}_n^{\mathrm{o}} \to \mathscr{B}_{n-1}^{\mathrm{o}}$ is the subalgebra generated by the classes x_{in}, which is in fact a free Lie algebra (see the remarks following the proof of Theorem 2.1). We identify the kernel of all of the ϕ^ℓ as a submodule of this free Lie algebra.

Definition 4.3. For $n > 1$, let $M_{d,n}$ be the submodule of of the degree d summand of $\mathscr{BT}_n^{\mathrm{o}}$ generated by brackets of the classes x_{in} such that each i from 1 to $n-1$ appears as an index. Let $M_{1,1} = \mathscr{BT}_1^{\mathrm{o}}$.

Theorem 4.4. *There is a spectral sequence which converges to* $\pi_*(Emb(I, \mathbb{R}^k \times I))$ *whose* E^1 *term is given by* $E^1_{-n,d(k-1)+1} = M_{d,n}$ *and whose* d^1 *is the restriction to this submodule of the* d^1 *of Corollary* 4.3.

Note that $M_{d,n} = 0$ for $d < n-1$, which leads to the following vanishing theorem.

Corollary 4.5. *In the spectral sequence of Theorem* 4.4, $E^1_{-p,q} = 0$ *if* $q < p(k-1) + 2 - k$.

It is interesting to note that while the modules $M_{d,n}$ may be defined purely in terms of the free Lie algebra (on $n-1$ generators), the boundary maps between them require extending the free Lie algebra to a configuration space Lie algebra. From the algebraic definition of d^1 it is not obvious that its restriction to $M_{d,n}$ maps to $M_{d,n+1}$.

Since computing the E^2 term amounts to computing the cohomology of the complexes $M_{d,*}$, as a warmup we will compute the rank of $M_{d,n}$, which we denote $R(d,n)$, and will show that $\chi(M_{d,*}) = 0$. Recall that the number of Hall trees of degree d with n symbols is equal by Lemma 3.1 to $(1/d)\sum_{j|d}\mu(j)n^{d/j}$. We may produce a basis of $M_{d,n}$ by first considering all brackets of degree d and throwing away ones in which fewer than n elements appear. We find that

$$R(d,n) = \frac{1}{d}\sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}\sum_{j|d}\mu(j)i^{d/j}.$$

We pause to define $S(d,n) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}i^d$, which are essentially Stirling numbers. There is a combinatorial interpretation of $S(d,n)$ as the number of surjections from a d element set onto an n element set (to verify this, count all set maps and subtract the non-surjections). Note as well that the $S(d,n)$ have a generating function, as

$$\sum_{m=0}^{\infty} S(m,n)\frac{x^m}{m!} = (\mathrm{e}^x - 1)^n.$$

Reordering the summations of $R(d,n)$ we find the following:

Proposition 4.6. $R(d,n) = (1/d)\sum_{j|d}\mu(j)S(d/j,n)$.

We may give $R(d,n)$ a combinatorial interpretation in line with this equality as the number of surjections of a d element set to an n element set which are not invariant under any cyclic permutation of the d element set, modulo cyclic permutations of the d element set. It would be interesting to find a bijection between such equivalence classes of surjections and a basis of $M_{d,n}$. Such a combinatorial interpretation would be particularly interesting for $M_{n,n}$ which, along with its $\sum_n$ action by permuting the letters, is known as $Lie(n)$ and arises in the calculus of functors approach to homotopy theory [1].

Theorem 4.7. *The Euler characteristic of $M_{d,*}$ is zero for $d>2$.*

Proof. The Euler characteristic of the complex $M_{d,*}$ is by definition $\sum_{\ell=1}^{d}(-1)^{\ell}R(d,\ell)$, which after applying Proposition 4.6, reversing the order of summation, and ignoring zero terms, is equal to

$$\frac{1}{d}\sum_{j|d}\mu(j)\sum_{\ell=1}^{d/j}(-1)^{\ell}S(d/j,\ell).$$

We claim that $\sum_{\ell=1}^{m}(-1)^{\ell}S(m,\ell)=(-1)^{m}$, which can be verified by computing the coefficient of $x^m/m!$ of $\sum_{p=0}^{m}(-1)^{p}(\mathrm{e}^{x}-1)^{p}$. Hence the Euler characteristic is equal to $(1/d)\sum_{j|d}\mu(j)(-1)^{d/j}$. Let $t(d)=d\chi(M_{d,*})=\sum_{j|d}\mu(j)(-1)^{d/j}$. Applying Möbius inversion we find $\sum_{j|d}t(j)=(-1)^{d}$. Computing that $t(1)=-1$ and $t(2)=2$, we see by induction that $t(d)=0$ if $d>2$, proving the theorem. □

5. Algorithms

In this section, we provide a detailed description of the methods used to compute the boundary operators in the complexes described above. These algorithms can be performed by hand for the complexes of small degree d, but are best implemented on the modern electronic computer otherwise.

Because the product of two Hall trees is not necessarily a Hall tree, one must have an algorithm which takes an arbitrary tree representing a free Lie algebra element, and expresses it as a linear combination of Hall trees. The proof that this algorithm terminates and produces the desired result is contained in the proof of Theorem 4.9 in [9].

Algorithm 5.1 (Hallification)**.** Given an integral linear combination of trees representing an element of $\mathscr{L}(A)$, this algorithm outputs a linear combination of Hall trees representing the same element of $\mathscr{L}(A)$.

1. If each tree appearing with a non-zero coefficient in the linear combination is Hall, terminate the algorithm and output the linear combination. Otherwise, choose t to be the first non-Hall tree appearing in the linear combination and proceed.
2. Find a subtree $s=[s_1,s_2]$ of t which is not Hall but whose children s_1 and s_2 are Hall. This can be achieved by a simple recursion, noting that the degree one trees (single letters) are Hall.
3. If $s_1=s_2$, then remove t from the linear combination and go to step (1).
4. If $s_1>s_2$, then switch s_1 and s_2 in t, multiply the coefficient of t by -1, and go to step (1).
5. We have $s_1<s_2$. In this case, s_1 cannot be a single letter, or else s would be Hall. So $s_1=[x,y]$. We must have $y<s_2$ again using the fact that s is not Hall. Replace t in the linear combination by the sum of two trees obtained by replacing $s=[[x,y],s_2]$ by $[[x,s_2],y]$ and $[x,[y,s_2]]$ respectively, and go to step (1).

The following algorithm uses the relations for $\mathscr{B}_n^{\mathrm{o}}$ from Definition 2.1 and the Jacobi identity to express elements of $\mathscr{B}_n^{\mathrm{o}}$ in a standardized form. It will be used in the computation of the boundary operator ∂^n in Algorithm 5.3 below.

Definition 5.1. We say that a bracket in the classes x_{ij} for $1 \leqslant i, j \leqslant n$ is *pure* if either all x_{ij} which appear are of the form x_{in} or none are of this form.

Algorithm 5.2 (Standard basis for $\mathscr{B}_n^{\mathrm{o}}$). Given an element x of $\mathscr{B}_n^{\mathrm{o}}$ expressed as a linear combination of brackets in the x_{ij}, this algorithm computes a linear combination of pure brackets also representing x.

1. If each bracket appearing with a non-zero coefficient in the linear combination is pure, terminate the algorithm and output the linear combination. Otherwise, choose t to be the first bracket in the linear combination which is not pure and proceed.
2. Find a smallest degree sub-bracket $s = [s_1, s_2]$ of t which is not pure. A simple recursion finds this sub-bracket.
3. If the degree of s is two, go to step (4), otherwise go to step (7).
4. Since the degree of s is two, we have $s_1 = x_{ij}$ and $s_2 = x_{\ell m}$ with either $j = n$ or $m = n$. If $j = n$, go to step (5) and if $m = n$, go to step (6).
5. If $i = \ell$, then replace t in the linear combination by a new bracket obtained from t by replacing $s = [x_{in}, x_{im}]$ with $[x_{mn}, x_{in}]$, using relation (4) in the definition of $\mathscr{B}_n^{\mathrm{o}}$. If $i = m$, then do the same thing, replacing $s = [x_{mn}, x_{\ell m}]$ with $[x_{\ell n}, x_{mn}]$ by the same relation. In all other cases, remove t from the linear combination (applying relation (3) in the definition of $\mathscr{B}_n^{\mathrm{o}}$). Start over at step (1).
6. If $i = \ell$, then replace t in the linear combination by a new bracket obtained from t by replacing $s = [x_{ij}, x_{in}]$ with $[x_{in}, x_{jn}]$. If $\ell = j$, then do the same thing, replacing $s = [x_{ij}, x_{jn}]$ with $[x_{jn}, x_{in}]$. In all other cases, remove t from the linear combination. Start over at step (1).
7. If the degree of s_1 is greater than one, say $s_1 = [x, y]$, we use the Jacobi identity to replace t in the linear combination by the sum of two brackets obtained from t by replacing the sub-bracket $s = [[x, y], s_2]$ by $[[s_2, y], x]$ and $[[x, s_2], y]$ respectively. If s_1 has degree one, then s_2 must have degree at least two, say $s_2 = [x, y]$, and we do the same thing, replacing $s = [s_1, [x, y]]$ by $[x, [s_1, y]]$ and $[y, [x, s_1]]$, respectively, and adding the results. In either case, start over at step (1).

A simple induction argument shows that this algorithm terminates and produces the desired result. Namely, we associate to a bracket t the pair (a, b) where a is the number of generators x_{ij} with $j < n$ appearing in t, and b is the degree of the smallest impure sub-bracket found in step (2). We order such pairs lexicographically, with the minimum $(0, 0)$ being achieved by pure brackets. At every step, this algorithm produces brackets whose associated pairs are less than that of the original. Steps (5) and (6), corresponding to $b = 2$, clearly reduce a. Step (7) leaves a unchanged but reduces b, since the sub-bracket $[x, y]$ of s which is initially pure becomes impure in all terms which occur after applying the Jacobi identity. Finally note that since b in such an associated pair is bounded by the degree of the bracket, there are only finitely many pairs less than a given one, so the algorithm must terminate after a finite number of recursive steps.

Note that the terms in the linear combination output by Algorithm 5.2 which do not involve any x_{in} can be run recursively through the algorithm as elements of $\mathscr{B}^{o}_{n-1}$, yielding the standard form claimed in the proof of Theorem 2.1.

The final algorithm is the heart of the calculation; it computes ∂^{ℓ} for $\ell = 0, \ldots, n$, exploiting the fact that these maps are Lie algebra homomorphisms. Observe that ∂^{n+1} is simply the natural inclusion of $\mathscr{BT}^{o}_{n}$ into $\mathscr{BT}^{o}_{n+1}$ and therefore requires no detailed description.

Algorithm 5.3 (Boundary operator)**.** Given a basis element t of $M_{d,n}$ (expressed as a Hall tree) and an integer ℓ between 0 and n, this algorithm computes $\partial^{\ell}(t)$ as a linear combination of degree d elements of $\mathscr{B}^{o}_{n+1}$ in the standard form given by Algorithm 5.2.

1. If the degree of t is greater than one, say $t = [t_1, t_2]$, then recursively call Algorithm 5.3 to compute $\partial^{\ell}(t_1)$ and $\partial^{\ell}(t_2)$. Set $\partial^{\ell}(t) = [\partial^{\ell}(t_1), \partial^{\ell}(t_2)]$ and proceed to step (2). If the degree of t is one, go to step (4).
2. If $\ell = n$, then use Algorithm 5.2 to express the answer $\partial^{\ell}(t)$ in standard form. Proceed to step (3).
3. Use Algorithm 5.1 to express $\partial^{\ell}(t)$ in terms of Hall trees. Terminate the algorithm and return $\partial^{\ell}(t)$.
4. If $\ell < n$, proceed to step (5), otherwise go to step (6).
5. Assume $t = x_{in}$. If $i < \ell$, set $\partial^{\ell}(t) = x_{i,n+1}$. If $i > \ell$, set $\partial^{\ell}(t) = x_{i+1,n+1}$. If $i = \ell$, set $\partial^{\ell}(t) = x_{i,n+1} + x_{i+1,n+1}$. Go to step (3).
6. Assume $t = x_{in}$. Set $\partial^{\ell}(t) = x_{in} + x_{i,n+1}$ and go to step (2).

An example of this algorithm is worked out by hand in the next section.

6. Results

In this section, we present some results of the computations described in the previous section. We will choose the gradings to correspond to the case $k = 3$, i.e. embeddings in $\mathbb{R}^4$.

First, we note that in the degree one case, we have $E^1_{-1,3} = \mathbb{Q}$, generated by y_1, $E^1_{-2,3} = \mathbb{Q}$ generated by x_{12}, and d^1 is an isomorphism. In degree two, the only non-zero entry is $E^1_{-3,5} = \mathbb{Q}$, generated by $[x_{13}, x_{23}]$, implying $E^2_{-3,5} = \mathbb{Q}$.

We proceed by working out the first non-trivial boundary operator $d^1 : E^1_{-3,7} \to E^1_{-4,7}$ by hand. These spaces are by definition $M_{3,3}$ and $M_{3,4}$. Bases are obtained by creating, with Algorithm 3.1, Hall bases for the free Lie algebra generated by $\{x_{13}, x_{23}\}$ (resp. $\{x_{14}, x_{24}, x_{34}\}$) and selecting the elements which have degree 3 and such that all possible values of i appear. It turns out that each space is two-dimensional; the first is generated by $[x_{13},[x_{13},x_{23}]]$ and $[[x_{13},x_{23}],x_{23}]$ and the second by $[x_{14},[x_{24},x_{34}]]$ and $[[x_{14},x_{34}],x_{24}]$. Algorithm 5.3 is straightforward for $\ell \neq 3$; in these cases we have

$$\partial^0[x_{13},[x_{13},x_{23}]] = [x_{24},[x_{24},x_{34}]],$$

$$\partial^0[[x_{13},x_{23}],x_{23}] = [[x_{24},x_{34}],x_{34}],$$

while

$$\begin{aligned}\partial^1[x_{13},[x_{13},x_{23}]] &= [x_{14}+x_{24},[x_{14}+x_{24},x_{34}]]\\ &= [x_{14}+x_{24},[x_{14},x_{34}]+[x_{24},x_{34}]]\\ &= [x_{14},[x_{14},x_{34}]]+[x_{14},[x_{24},x_{34}]]+[x_{24},[x_{14},x_{34}]]+[x_{24},[x_{24},x_{34}]]\\ &= [x_{14},[x_{14},x_{34}]]+[x_{14},[x_{24},x_{34}]]-[[x_{14},x_{34}],x_{24}]+[x_{24},[x_{24},x_{34}]]\end{aligned}$$

and

$$\begin{aligned}\partial^1[[x_{13},x_{23}],x_{23}] &= [[x_{14}+x_{24},x_{34}],x_{34}]\\ &= [[x_{14},x_{34}]+[x_{24},x_{34}],x_{34}]\\ &= [[x_{14},x_{34}],x_{34}]+[[x_{24},x_{34}],x_{34}],\end{aligned}$$

where the last line in the first case comes from an application of Algorithm 5.1 for the free Lie algebra over $\{x_{14},x_{24},x_{34}\}$. Similarly we have for $\ell = 2$

$$\begin{aligned}\partial^2[x_{13},[x_{13},x_{23}]] &= [x_{14},[x_{14},x_{24}]]+[x_{14},[x_{14},x_{34}]]\\ \partial^2[[x_{13},x_{23}],x_{23}] &= [[x_{14},x_{24}],x_{24}]+[[x_{14},x_{24}],x_{34}]+[[x_{14},x_{34}],x_{24}]+[[x_{14},x_{34}],x_{34}]\\ &= [[x_{14},x_{24}],x_{24}]+[[x_{14},x_{34}],x_{34}]\\ &\quad +2*[[x_{14},x_{34}],x_{24}]+[x_{14},[x_{24},x_{34}]],\end{aligned}$$

where again the last line comes from Algorithm 5.1. Finally, as noted above, ∂^4 is the natural inclusion

$$\partial^4[x_{13},[x_{13},x_{23}]] = [x_{13},[x_{13},x_{23}]],$$

$$\partial^4[[x_{13},x_{23}],x_{23}] = [[x_{13},x_{23}],x_{23}].$$

The case $\ell=3$ is much more computationally taxing, as it requires the use of Algorithm 5.2

$$\begin{aligned}\partial^3[x_{13},[x_{13},x_{23}]] &= [x_{13}+x_{14},[x_{13}+x_{14},x_{23}+x_{24}]]\\ &= [x_{13},[x_{13},x_{23}]]+[x_{14},[x_{14},x_{24}]]+[x_{24},[x_{34},x_{14}]]\\ &\quad +[x_{14},[x_{34},x_{24}]] && \text{(Alg. 5.2)}\\ &= [x_{13},[x_{13},x_{23}]]+[x_{14},[x_{14},x_{24}]]\\ &\quad +[[x_{14},x_{34}],x_{24}]-[x_{14},[x_{24},x_{34}]], && \text{(Alg. 5.1)}\end{aligned}$$

Table 1
E^1 term for $k=3$

$\mathbb{Q}^{120}$	$\mathbb{Q}^{300}$	$\mathbb{Q}^{260}$	$\mathbb{Q}^{89}$	$\mathbb{Q}^{9}$			13
							12
	$\mathbb{Q}^{24}$	$\mathbb{Q}^{48}$	$\mathbb{Q}^{30}$	$\mathbb{Q}^{6}$			11
							10
		$\mathbb{Q}^{6}$	$\mathbb{Q}^{9}$	$\mathbb{Q}^{3}$			9
							8
			$\mathbb{Q}^{2}$	$\mathbb{Q}^{2}$			7
							6
				$\mathbb{Q}$			5
							4
					$\mathbb{Q}$	$\mathbb{Q}$	3
−7	−6	−5	−4	−3	−2	−1	

$$\begin{aligned}\partial^3[[x_{13},x_{23}],x_{23}] &= [[x_{13}+x_{14},x_{23}+x_{24}],x_{23}+x_{24}]\\ &= [[x_{13},x_{23}],x_{23}]+[[x_{14},x_{24}],x_{24}]+[[x_{14},x_{34}],x_{24}]\\ &\quad +[[x_{24},x_{34}],x_{14}] \qquad \text{(Alg. 5.2)}\\ &= [[x_{13},x_{23}],x_{23}]+[[x_{14},x_{24}],x_{24}]+[[x_{14},x_{34}],x_{24}]\\ &\quad -[x_{14},[x_{24},x_{34}]]. \qquad \text{(Alg. 5.1)}\end{aligned}$$

Since $d^1=\sum_i(-1)^i\partial^i$, we have from the above calculations that

$$d^1[x_{13},[x_{13},x_{23}]]=0$$

$$d^1[[x_{13},x_{23}],x_{23}]=2*[x_{14},[x_{24},x_{34}]]+[[x_{14},x_{34}],x_{24}].$$

and so the matrix for the boundary operator with respect to our chosen bases is given by

$$\begin{pmatrix}0 & 2\\ 0 & 1\end{pmatrix}.$$

We conclude that the boundary operator has rank one, and so $E^2_{-3,7}\cong E^2_{-4,7}\cong\mathbb{Q}$. Further (computer) calculations yield the E^1 and E^2 terms for k odd given in Tables 1 and 2.

These low-dimensional computations do not reveal any regular behavior. Note that, as allowed because the Euler characteristic of the rows is zero, some rows vanish while most do not. Note as well that there is no additional vanishing along the edge of the vanishing line of Corollary 4.5.

All of the classes in Table 2 survive to E^∞ except perhaps those in bidegrees $(-6,13)$ and $(-3,11)$ which could support a d^3 differential.

Table 2
E^2 term for $k = 3$

$\mathbb{Q}$	$\mathbb{Q}$					13
						12
	$\mathbb{Q}^2$	$\mathbb{Q}$		$\mathbb{Q}$		11
						10
						9
						8
			$\mathbb{Q}$	$\mathbb{Q}$		7
						6
				$\mathbb{Q}$		5
						4
						3
-7	-6	-5	-4	-3	-2	

Theorem 6.1. *There are non-trivial classes in* $\pi_n(Emb(I, \mathbb{R}^3 \times I))$ *for* $n = 2, 3, 4, 5, 6$.

It would be interesting to find explicit spherical families of embeddings which represent these classes. One expects the evaluation map

$$\Delta^n \times Emb(I, \mathbb{R}^k \times I) \to F(\mathbb{R}^k \times I, n) \times (S^k)^n$$

to play a central role in relating these homotopy groups to those of $F(\mathbb{R}^k \times I, n) \times (S^k)^n$ which appear in our spectral sequence.

We conclude with a brief description of some of the methods used to verify the computer calculations (beyond merely computing examples by hand and comparing with the computer output, which was done extensively). Algorithm 3.1 was checked by an independent function which verified that the generated trees were Hall, and checked the number of elements in the resulting Hall set against the dimension count given by Lemma 3.1. It was verified in the course of computing the E^2 term in Table 2 that $d^2 = 0$ for each of the chain complexes comprising E^1. A similar mathematical fact which was not hard-coded into the application is that the image of $M_{d,n}$ under d^1 lands in $M_{d,n+1}$ despite the fact that this is not the case for the individual homomorphisms ∂^ℓ. The ranks of the boundary operators were verified using the linear algebra capabilities of a symbolic mathematics package (Maple). Finally, a nice check of the system as a whole was provided by varying the algorithm for generating Hall sets (noting the choices made in Algorithm 3.1) and verifying that the ranks of all boundary operators remained unchanged.

7. Further work

In further work [10] we will investigate the case of k even, which includes the case of classical knots. Though in the case of classical knots the spectral sequences of [11] do not necessarily converge, one can use those methods to produce knot invariants, which we show are of finite type. In particular, an optimistic view of rational homotopy theory predicts that the module of classes along the vanishing line of our spectral sequence (which for $k=2$ is the anti-diagonal) is isomorphic to the module of primitives in the Hopf algebra of finite type invariants [2]. To prove such a conjecture would involve relating the combinatorics of configuration space Lie algebras to those of Feynman diagrams, which could give a satisfactory explanation in terms of algebraic topology of the appearance of Feynman diagrams in the study of knots.

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