LORENTZ SPACETIMES OF CONSTANT CURVATURE

GEOFFREY MESS

Contents

1. Introduction

A 2+1 dimensional flat spacetime M is a connected 3-manifold with a flat Lorentz metric, or equivalently a manifold with charts $f_\alpha : U_\alpha \to \mathbb{R}^{2+1}$ with transition functions $f_\alpha \circ f_\beta^{-1}$ given by a Lorentz isometry on $f_\beta(U_\alpha \cap U_\beta)$ (assuming $U_\alpha \cap U_\beta$ is connected). In the terminology of geometric structures, M has a $(\mathbf{ISO}(2,1), \mathbb{R}^{2+1})$-structure. Here $\mathbf{ISO}(2,1)$ is the automorphism group of $(\mathbb{R}^{2+1}, -dt^2 + dx^2 + dy^2)$. It is a semidirect product $\mathbf{O}(2,1) \ltimes \mathbb{R}^{2+1}$ where $\mathbb{R}^{2+1}$ is the subgroup of translations. We restrict attention to oriented orthochronous spacetimes; i.e. the transition functions actually lie in $\mathbf{SO}(2,1)_0 \ltimes \mathbb{R}^{2+1}$. Any noncompact connected orientable 3-manifold immerses in $\mathbb{R}^{2+1}$ by the immersion theorem [1], so some restriction is needed.

Witten [2] proposes the problem of classifying flat spacetimes which are topologically of the form $S \times (-1, 1)$ and contain $S \times \{0\}$ as a spacelike hypersurface. Any such spacetime contains many open sets satisfying the same topological condition. Thus additional geometrical hypotheses, or else some natural equivalence relation on spacetimes are necessary for a reasonable classification. Proposition 6, 7, 8, 15, 26 show that it is natural to consider the class of manifolds which are domains of dependence (definition 4). Witten also proposes the problems of classifying spacetimes which are of constant curvature and are neighbourhoods of a closed spacelike hypersurface, and more generally, conformally flat Lorentzian spacetimes. The domains of dependence are spacetimes which are maximal with respect to the property that there is a closed spacelike hypersurface through each point. In the case of flat spacetimes, we show in section 2, 3, 4, 5 that for a given genus, all domains of dependence form two families, each parametrized by the cotangent bundle of Teichmüller space.

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All the spacetimes in one family have an initial singularity and are future complete while the other family differs only by time reversal.

In the case of anti de Sitter spacetimes, that is, those which have constant curvature -1, we obtain a complete classification in section 7. There is a natural family of maximal domains of dependence, parametrized by the product of two copies of Teichmüller space, such that given any closed spacelike hypersurface in an anti de Sitter spacetime, a neighbourhood of the hypersurface embeds as an open subset of a unique member of the natural family of spacetimes. These spacetimes are Lorentzian analogues of the hyperbolic manifolds corresponding to quasifuchsian groups. There is a convex core and a consideration of the bending lamination on the two boundary components of the convex core leads to a new proof of Thurston's theorem on the existence of a left earthquake and a right earthquake joining any two points in Teichmüller space. In fact this new proof is essentially Thurston's second (and elementary) proof of the existence of earthquakes, interpreted geometrically in anti de Sitter space. The spacetimes in this family have both an initial and a final singularity. These singularities are qualitatively similar to the initial singularities of the flat spacetimes.

On the other hand, in the case of de Sitter spacetimes, we only give a conjectural statement of a complete classification. In section 6 we construct a family of de Sitter spacetimes, each of which is foliated by closed spacelike hypersurfaces, which conjecturally is the solution to the problem. Each spacetime corresponds to a projective structure on a Riemann surface at future infinity. There is another family of spacetimes which have a Riemann surface at past infinity. Let us remark that most spacetimes in this family fail to have the property that their universal covers embed in the model space (that is, de Sitter space). This is in contrast to the situation for flat Lorentzian manifolds and for anti de Sitter manifolds, where the universal covers are convex regions in $2+1$-dimensional Minkowski space or anti de Sitter space. Furthermore, in the case of de Sitter space there is always an infinite discrete set of spacetimes corresponding to a given representation of the fundamental group, while in the cases of curvature -1 or 0 the representation determines the spacetime essentially uniquely.

We have nothing to say about conformally flat Lorentzian manifolds, nor about the interesting question of foliating the manifolds we consider by surfaces of constant mean curvature of constant curvature; however let us mention [52], cf. section 6.

Y. Carrière [5] proved a beautiful theorem on the completeness of closed flat Lorentzian manifolds. In section 8, we generalize this to the case of compact Lorentzian manifolds with spacelike boundary. Thus a flat Lorentzian manifold with boundary is topologically a product, and is foliated by closed spacelike hypersurfaces. This result generalizes to Lorentzian manifolds of constant negative curvature. Probably it also holds in the case of positive curvature, but we have not shown this. We also make some remarks about the problem of classifying flat Lorentzian manifolds containing a closed spacelike hypersurface in the physical case of 3+1 dimensions. While in 2+1 dimensions Einstein's equations require the spacetime to be flat (or of constant curvature if the cosmological constant is nonzero) in 3+1 dimensions the spacetimes of constant curvature are very special, but nonetheless interesting, solutions to Einstein's equations.

Witten's paper [2] is concerned with the solution of the quantum theory of general relativity. The definition of the Hilbert space of the theory depends on knowing the space of classical solutions. In Witten's theory, classical quantities such as the traces of holonomies of elements of the fundamental group are observables, but there are also remarkable new observables which (unlike the holonomy) distinguish between homotopic, but nonisotopic knots. Other references to the physical literature are given in [3, 4].

Recall that given a (G, X)-manifold M, there is a *development map* $\mathsf{dev} : \widetilde{M} \to X$ where we use $\widetilde{}$ to denote universal covers. The map dev is a submersion and satisfies $\mathsf{dev}\,(\gamma\cdot x) = \rho(\gamma)\mathsf{dev}\,(x)$ for each $\gamma \in \pi_1 M$ and a homomorphism $\rho : \pi_1 M \to G$ (called the *holonomy map*), determined up to a conjugacy in G. The development dev is characterized by the fact that the lift of the (G, X)-structure to $\widetilde{M}$ is given by charts $\mathsf{dev}\,|_{V_\alpha} V_\alpha \to X$.

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2. Fuchsian holonomy

Suppose M contains a closed spacelike hypersurface S. At each point $p \in S$, let $\mathbf{n}$ be the future pointing unit timelike normal vector. To a unit tangent vector $\mathbf{v}$ at p, assign the null vector $\mathbf{n}+\mathbf{v}$. This identifies the unit tangent bundle of S with the projectivized bundle of null directions. Let $q : \mathbf{ISO}(2,1) \to \mathbf{SO}(2,1)_0$ be the quotient map. We identify $\mathbf{SO}(2,1)_0$ with $\mathbf{PSL}(2,\mathbb{R})$.

Then the unit tangent bundle of S is the $\mathbb{RP}^1$-bundle with structure group $\mathbf{SO}(2,1)_0$ associated with the flat principal $\mathbf{SO}(2,1)_0$-bundle over S given by

$$(\widetilde{S} \times \mathbf{SO}(2,1)_0)/\pi_1 S$$

where $\gamma \in \pi_1 S$ acts on $\widetilde{S} \times \mathbf{SO}(2,1)_0$ by $\gamma \cdot (s, g) = (s \cdot \gamma, q(\rho(\gamma)) \cdot g)$. So the Euler class of the homomorphism $q\rho : \pi_1 S \to \mathbf{SO}(2,1)_0$ is the Euler class of the tangent bundle, that is, $2-2g$ if S is oriented so the positive normal to S is future pointing. If $g > 1$, then by Goldman's theorem [6] $q\rho$ maps $\pi_1 S$ isomorphically to a discrete cocompact subgroup of $\mathbf{SO}(2,1)_0$. So:

Proposition 1. *If $g > 1$ then $q\rho : \pi_1 S \to \mathbf{SO}(2,1)_0$ is injective and $q\rho(\pi_1 S)$ is a discrete cocompact subgroup of $\mathbf{SO}(2,1)_0$.*

Proposition 1 was conjectured by Witten in [2]. It is possible to avoid Goldman's theorem (cf. the end of section 8) but we will use an extension of the present argument when considering anti de Sitter manifolds in section 7.

For the reader's convenience we give a short exposition of the Milnor-Wood inequality [7, 8] and Goldman's theorem. Suppose $E \to T$ is an S^1-bundle over a closed surface T with structure group the orientation preserving homeomorphisms of S^1 with the discrete topology. In other words the transition functions for the bundle are locally constant homeomorphisms of S^1. Let g be the genus of T. Regard T as a $4g$-gon D, together with $2g$ rectangles (the 1-handles) attached to represent the generators $(a_1, a_2, \ldots, a_{2g-1}, a_{2g})$, and a second $4g$-gon D'. Let $h : \pi_1 T \to \mathsf{Homeo}^+(S^1)$ be the holonomy map. (We regard $\mathsf{Homeo}^+(S^1)$ as acting on the *right* of S^1.) Regard S^1 as $\mathbb{R}/\mathbb{Z}$, and consider $H = \mathsf{Homeo}^+(S^1)$ as the quotient of the subgroup $H^+ \subset \mathsf{Homeo}^+(\mathbb{R})$ defined by $f(x+1) = f(x)+1$ by its center, the group generated by $z(x) = x+1$. For each generator, choose a lift $h^*(a_i)$ to H^+, such that $0 \le 0 \cdot h^*(a_i) < 1$ and $h^*(a_i^{-1}) = (h^*(a_i))^{-1}$.

Then there is a section s of the S^1-bundle over $D \cup$ (the 1-handles) which (trivializing the bundle over D) equals 0 and $0 \cdot h(a_i)$ over the initial and final sides of the ith 1-handle and is locally constant along each 1-handle. The section is extended radially over D, so $s = 0$ at the center of D and is linear along each radius of D. $\prod_{i=1}^{g} [h^*(a_{2i-1}), h^*(a_{2i})](x) = x - e$ for some integer e. Choose a constant section say s' over the disc D'. The Euler class of the bundle E is the obstruction to extending s over D'. It equals $e \in \mathbb{Z} \cong H^2(T, \mathbb{Z})$, choosing the generator so $\prod_{i=1}^{g} [a_{2i-1}, a_{2i}]$ represents the oriented boundary of $D \cup$ (the 1-handles). Then $0 \cdot h^*(a_1) < 1$, $0 \cdot h^*(a_1 a_2) < 2$, $0 \cdot h^*(a_1 a_2 a_1^{-1}) < 2$, $0 \cdot h^*(a_1 a_2 a_1^{-1} a_2^{-1}) < 2$, $0 \cdot h^*(a_1 a_2 a_1^{-1} a_2^{-1} a_3) < 3$ and so on.

It follows that $-e \le 2g - 1$. Similarly $-e \ge 1 - 2g$. These inequalities can be improved: The genus of a k-sheeted covering space of T is $g' = k(g-1)+1$. The Euler number of the pullback of the bundle to the covering space is $k \cdot e$. So $|e| \le 2g - 2$, if $g > 1$, and $e = 0$ if $g = 1$. All values in this range are attained; the value $2 - 2g$ is attained by the flat S^1-bundle with structure group $\mathbf{PSL}(2, \mathbb{R})$ (acting on $S^1 = \mathbb{RP}^1$) over a hyperbolic surface obtained by identifying the unit tangent vectors at each point in the hyperbolic plane with the circle at infinity. (Of course this S^1-bundle admits no flat structure with structure group S^1.) Benzécri's theorem [9, 10] follows: A closed surface of negative Euler characteristic admits no flat linear connection (not necessarily torsion-free), and therefore no affine structure. For the Euler number of the projective tangent bundle is $4 - 4g$. More generally, an $\mathbb{R}^2$-bundle with a flat $\mathbf{GL}(2, \mathbb{R})$ connection has Euler number at most $g - 1$ in absolute value.

Now consider the components of the space of representations of a surface group $\pi_1 T$, where T has genus $g > 1$, in $\mathbf{PSL}(2, \mathbb{R})$. A representation ρ defines a flat S^1-bundle over T with structure group $\mathbf{PSL}(2, \mathbb{R})$ and Euler number denoted by $e(\rho)$. If ρ is discrete and faithful this is the tangent bundle (up to change of orientation) so $|e| = 2g - 2$. Conversely Goldman shows in [6] that if $|e(\rho)| = 2g - 2$ then ρ is discrete and faithful, so $e(\rho) = 2g - 2$ defines the component of the space of representations which consists of discrete and faithful representations such that the orientation of T agrees with that of $\mathbb{H}^2/\rho(\pi_1 T)$ and so can be identified with $\mathbf{PSL}(2, \mathbb{R}) \times \mathsf{Teich}(T)$ where $\mathsf{Teich}(T)$ is Teichmüller space. Suppose ρ is a representation with $e(\rho) = 2 - 2g$. If ρ is not irreducible, it is solvable from which it would follow that $e(\rho) = 0$. So assume that ρ is irreducible. There is then a $6g-3$-dimensional neighbourhood of ρ in the space of homomorphisms and no function $\mathsf{tr}^2(\rho(\gamma))$ is locally constant on the neighbourhood. Suppose $\gamma \in \pi_1 T - \{1\}$ and $\rho(\gamma)$ is an elliptic element of $\mathbf{PSL}(2, \mathbb{R})$, possibly the identity. By a small change in ρ, which doesn't change the Euler number, we attain that $\rho(\gamma)$ has finite order say n. By [11], there is a finite sheeted cover T' of T on which γ^n is represented by a simple closed curve C. We have $e(B') = \chi(T')$ where B' is the flat S^1-bundle on T' determined by $\rho|_{\pi_1(T')}$. But surgery on C gives a 3-manifold M such that $\partial M = T' \cup T''$ where $\chi(T'') = \chi(T') + 2$ and the bundle extends to M because the holonomy of C is trivial. T'', with the orientation inherited from M, is homologous to $-T$. This contradicts the Milnor-Wood inequality. So ρ is faithful and $\mathrm{im}(\rho)$ has no elliptics, and considering the closed subgroups of $\mathbf{PSL}(2, \mathbb{R})$ (cf. [12]) shows that $\mathrm{im}(\rho)$ is discrete. R. Brooks has informed me that this proof is given in [39]. Goldman's proof using foliation is more geometric and the technique is of much more general use.

A closed spacelike hypersurface S cannot have $g = 0$. Otherwise we would have a spacelike immersion $S \to \mathbb{R}^{2+1}$. The projection of S onto $\mathbb{R}^2$ is then an immersion, which is absurd. We let $g > 1$ in the following proposition, returning to the case $g = 1$ below.

Proposition 2. *Given a discrete and faithful $f : \pi_1 S \to \mathbf{SO}(2,1)_0$, there is a flat Lorentz spacetime M with $M = S \times (0, \infty)$ and holonomy $\rho : \pi_1 M \to \mathbf{ISO}(2,1)$ equal to f.*

Proof. $f(\pi_1 S)$ acts properly discontinuously on each of $L_+ = \{(x,y,t) : t^2 > x^2 + y^2, t > 0\}$ and $L_- = \{(x,y,t) : t^2 > x^2 + y^2, t < 0\}$. Take M to be $L_+/f(\pi_1 S)$ or $L_-/f(\pi_1 S)$. (This has been well known at least since [13].) □

Note that M can be chosen either future complete or past complete. (An orthochronous Lorentz manifold is future complete if every future pointing timelike or null geodesic segment can be extended to all positive values of an affine parameter; equivalently, the developed geodesic extends to a future pointing ray in $\mathbb{R}^{2+1}$. See section 4.)

3. Realization of holonomy homomorphisms

Fix a homomorphism $f : \pi_1 S \to \mathbf{SO}(2,1)_0$. Suppose $\rho : \pi_1 S \to \mathbf{ISO}(2,1)$ is a homomorphism with $q\rho = f$. Then $\rho(\gamma)x = f(\gamma)x + t_\gamma$ for some $t : G \to \mathbb{R}^{2+1}, \gamma \mapsto t_\gamma$, where t is a 1-cocycle, that is, $t_{\alpha\beta} = t_\alpha + f(\alpha)t_\beta$. If there exists some v such that $t_\gamma = v - f(\gamma)v$, then $\rho(\gamma)$ differs from f only by conjugation by the translation $x \mapsto x + v$. The quotient $H^1(\pi_1 S, \mathbb{R}^{2+1})$ of the cocycles by the coboundaries corresponds to the space of conjugacy classes of representations $\rho : \pi_1 S \to \mathbf{ISO}(2,1)$ such that $q\rho = f$. We fix an identification of the quotient space with a subspace of the space of cocycles.

Proposition 3. *Given $f : \pi_1 S \to \mathbf{SO}(2,1)_0$ and $M = L_+/f(\pi_1 S)$ as in proposition 2, and any bounded neighbourhood U of 0 in $H^1(\pi_1 S, \mathbb{R}^{2+1})$ there exists C such that, letting $M_C = \{(x,y,t) : -t^2 + x^2 + y^2 \leq -C\}/f(\pi_1 S)$, there exists a family of flat Lorentz metrics on M_C parametrized by U such that the spacetime $M_C(u)$ corresponding to $u \in U$ has holonomy $\rho : \pi_1 M_C \to \mathbf{ISO}(2,1)$ such that $q\rho = f$ and $\rho(\gamma)x = f(\gamma)x + t_\gamma(u)$ where $t_\gamma(u)$ is a cocycle representing $u \in H^1(\pi_1 S, \mathbb{R}^{2+1})$. (We say that $M_C(u)$ represents u.) Moreover, the spacetimes $M_C(u)$ are future complete and S is locally strictly convex in each $M_C(u)$.*

Proof. Consider the spacetime $M' = \{(x,y,t) : t > 0, -1 \geq -t^2 + x^2 + y^2 \geq -2\}/f(\pi_1 S)$. By the Thurston-Lok holonomy theorem [14, 15, 16, 17], there is a neighbourhood U_0 of 0 in $H^1(\pi_1 S, \mathbb{R}^{2+1})$ and a family of $(\mathbf{ISO}(2,1), \mathbb{R}^{2+1})$-structures on M' parametrized by U_0 and such that the holonomy $\rho : \pi_1(M'(u)) \to \mathbf{ISO}(2,1)$ represents u.

Now choose $K > 0$ so that $U \subset K \cdot U_0$. The parametrized development map $\mathsf{dev} : \widetilde{M'} \times U_0 \to \mathbb{R}^{2+1}$ satisfies

$$\mathsf{dev}\,(\gamma \cdot x, v) = f(\gamma) \cdot x + t_\gamma(v).$$

Now let $M'' = K \cdot M' = \{(x,y,t) : t > 0, -K \geq -t^2 + x^2 + y^2 \geq -2K\}/f(\pi_1(S))$ and define $\mathsf{dev}' : \widetilde{M''} \times K \cdot U_0 \to \mathbb{R}^{2+1}$ by $\mathsf{dev}'(K \cdot x, K \cdot u) = K \cdot \mathsf{dev}\,(x, u)$. Then

$$\mathsf{dev}'(\gamma x, u) = K \cdot \mathsf{dev}\,(\gamma x/K, u/K) = K(f(\gamma)x/K + t_\gamma(u/K)) = f(\gamma)x + t_\gamma(u).$$

So dev' is the parametrized developing map of a family of flat Lorentz structures. We observe that if U_0 is taken sufficiently small, then $T = \{-t^2 + x^2 + y^2 = -1.5\}/f(\pi_1 S)$ remains locally strictly convex and spacelike in the Lorentz structure determined by any $u \in U_0$. So $\mathsf{dev}\,(\widetilde{T}, u)$ is a locally convex complete surface. Later we will need the following lemma:

Lemma 1. *Suppose $i : F \to \mathbb{R}^{2+1}$ is a spacelike immersion of a connected surface F and F is complete in the induced metric. Then F is the graph of some function $h : \mathbb{R}^2 \to \mathbb{R}$, and in particular is embedded.*

Proof. The projection $p : F \to \mathbb{R}^2$ given by $(i_1(s), i_2(s), i_3(s)) \to (i_1(s), i_2(s))$ is a distance increasing submersion since $i(F)$ is spacelike. Since F is complete, p is a covering; F is connected so $i(F)$ is a graph. □

By Lemma 1, $S_u = \mathsf{dev}\,(\widetilde{T}, u)$ is a convex surface. Given any point p in the future of S_u, there is a unique point $\pi(p)$ on S_u such that the proper time $\tau(p,q) = \sqrt{-(p-q)\cdot(p-q)}$ is maximized when $q = \pi(p)$. The point p lies on the normal to S_u at $\pi(p)$. $\pi(p)$ is unique and depends continuously on p because S_u is strictly convex. Now $p \to (\pi(p), \tau(p, \pi(p)))$ equivariantly identifies the future of S_u with $S_u \times [0, \infty)$.

So there is a family of flat Lorentz structures on $\{(x,y,t) : -t^2 + x^2 + y^2 < -1\}/f(\pi_1 S)$ parametrized by U. All are future complete. □

We have determined all possible holonomies $\rho : \pi_1 S \to \mathbf{ISO}(2,1)$ and have a family of future complete spacetimes parametrized by this set.

4. Standard spacetimes

Proposition 4. *Let M be a flat Lorentz spacetime. Suppose M contains a closed spacelike surface S of genus $g > 1$. Then i) development embeds $\widetilde{S}$ in $\mathbb{R}^{2+1}$ and ii) the coordinate function t is a proper function on $\widetilde{S}$.*

Proof. *i)* is by lemma 1. Future pointing unit normals in $\mathbb{R}^{2+1}$ can be identified with the hyperbolic plane $\mathbb{H}^2$. Let $p : \widetilde{S} \to S$ be the projection from the universal covering space. Let $T : \widetilde{S} \to \mathbb{H}^2$ be the map such that $T(s)$ is the future pointing unit normal at $\mathsf{dev}\,(s)$. There is an induced map $T : S \to \mathbb{H}^2/q\rho(\pi_1 S)$. By proposition 1, $\mathbb{H}^2/q\rho(\pi_1 S)$ is a surface of the same genus g as S. By construction the tangent bundle of S is pulled back from $\mathbb{H}^2/q\rho(\pi_1 S)$. Since the Euler number of the tangent bundle of S is $2-2g$, $T : S \to \mathbb{H}^2/q\rho(\pi_1 S)$ has degree $+1$. T induces an isomorphism of fundamental groups so $T : \widetilde{S} \to \mathbb{H}^2$ is proper and has degree 1. Because $T : \widetilde{S} \to \mathbb{H}^2$ is proper, the normal $a(s)dt + b(s)dx + c(s)dy$ satisfies $|a(s)| \to \infty$ as $s \to \infty$ on $\widetilde{S}$. Also, $a^2 = b^2 + c^2 + 1$. So

$$dt = -\frac{b(s)}{a(s)}dx - \frac{c(s)}{a(s)}dy.$$

and so $\|dt\| \to 1$ as $s \to \infty$, where $\| \quad \|$ denotes the length in the induced Riemannian metric on $\widetilde{S}$. Furthermore, since T has degree 1, for any point $x \in \mathbb{H}^2$ and any sufficiently large simple closed curve C in $\widetilde{S}$, $T(C)$ has winding number 1 about x. Consider $\widetilde{S}$ as the graph of $t = t(x, y)$, we have that the map $w : S^1 \to S^1$ given by $w(\theta) = (b(s)^2 + c(s)^2)^{-\frac{1}{2}}(-b(s), -c(s))$, where $s = (R\cos\theta, R\sin\theta, t(R\cos\theta, R\sin\theta))$, has degree 1 for R sufficiently large. Replace t by a function u which equals t outside a compact set and has exactly one critical point, which is nondegenerate. This is possible since $\mathsf{degree}(w) = 1$. Furthermore P is a maximum or minimum rather than a saddle, since $\mathsf{degree}(w) = 1$. Without loss of generality assume P is a minimum. Let $B = u^{-1}\{\epsilon\}$ for some small ϵ. Then the gradient flow gives a submersion $F : B \times [0, \infty) \to \widetilde{S} - u^{-1}([0, \epsilon))$. Since $\|du\|$ is bounded below, F is a 1-1 covering and $|t| \to \infty$ as $s \to \infty$ on $\widetilde{S}$. □

Corollary 1. *If a spacetime M contains a closed spacelike surface S of negative Euler characteristic then S has an isochronous neighbourhood in M.*

Proof. Otherwise we would have both $t \to \infty$ and $t \to -\infty$ on $\mathsf{dev}\,\widetilde{S}$. □

Definition 1. *A flat Lorentz spacetime M such that $\pi_1 M$ is isomorphic to $\pi_1 S$ where S is a closed surface of negative Euler characteristic is a* standard spacetime *if (possibly after a change in time orientation) M is the quotient of the future in $\mathbb{R}^{2+1}$ of a complete spacelike strictly convex surface $\widetilde{S}$ by a group of Lorentz isometries acting cocompactly on $\widetilde{S}$.*

Given a flat Lorentz spacetime M containing a closed spacelike surface S, we may assume $t \to \infty$ on $\widetilde{S}$ (otherwise replace t by $-t$). We say S is "future directed".

Proposition 5. *Suppose M is a flat Lorentz manifold containing a closed spacelike future directed surface S of genus $g > 1$. There exists a standard spacetime M' containing a future directed strictly convex surface S' of genus g and such that S and S' have the same holonomy. Furthermore $\widetilde{S'}$ can be chosen to lie in the future of $\widetilde{S}$.*

Proof. By propositions 1 and 3 there exists a standard spacetime M' containing a strictly convex surface S' such that (fixing some identification between $\pi_1 S$ and $\pi_1 S'$) the holonomy homomorphisms from $\pi_1 S$ and $\pi_1 S'$ are equal. Replace S' by $S'(K)$, the surface of points which lie on the future pointing normals to S' at proper time K. S is compact, so there exists K sufficiently large such that for each point p in $\widetilde{S}$ some point of $\widetilde{S'(K)}$ lies in the future of p. Then $\widetilde{S'(K)}$ lies entirely in the future component of $\mathbb{R}^{2+1} - \widetilde{S}$, because no timelike line joins two points in $\widetilde{S'}$. □

Proposition 6. *If M is a spacetime with a closed spacelike surface S of negative Euler characteristic such that $\pi_1 S = \pi_1 M$, then there is a spacetime M'' containing a neighbourhood of S in M and containing a standard spacetime M'.*

Proof. We may assume $\widetilde{S}$ is future directed. Let $\widetilde{S'}$ be a spacelike convex surface such that for each point s of $\widetilde{S}$ some point of $\widetilde{S'}$ is in the future of s and $\widetilde{S'}$ is also invariant under $\rho(\pi_1 S)$. $\widetilde{S'}$ together with its future is our standard spacetime M'. It suffices to show that $\rho(\pi_1 S)$ acts properly discontinuously on the region R between the disjoint spacelike surfaces $\widetilde{S}$ and $\widetilde{S'}$; we then adjoin $R/\rho(\pi_1 S)$ to M' and thicken the resulting manifold slightly into the past of S. (See, for example, [16] for a formal discussion of thickening a manifold with a geometric structure.) Define a map $h : \widetilde{S} \times [0, 1] \to R$ by $(s, u) \to (1 - u)s + u\phi(s)$ where $\phi(s)$ is the intersection of the future pointing normal to $\widetilde{S}$ at s with $\widetilde{S}'$. h is proper: A compact subset K of R lies in a region $t \leq C$ for some C. So $h^{-1}K \subset \widetilde{S}$ contains the set $\{s \in \widetilde{S} : t \leq C\}$. By proposition 4, A is compact. Now to see $\rho(\pi_1 S)$ acts properly discontinuously on R, let $K \subset R$ be compact. Then for all but finitely many $g \in \rho(\pi_1 S)$, $g \cdot h^{-1}K \cap h^{-1}K = \emptyset = h^{-1}gK \cap h^{-1}K$ so $gK \cap K = \emptyset$. $\rho(\pi_1 S)$ is torsion free, so R covers $R/\rho(\pi_1 S)$. □

Since $\rho(\pi_1 S)$ acts properly discontinuously on R, and t is a proper function on R which is bounded below, $R/\rho(\pi_1 S)$ contains no closed timelike curves. By a standard argument [18], $R = S \times [0,1]$ where $S \times \{0\} = S$, $S \times \{1\} = S'$, and each curve $\{s\} \times [0,1]$ is timelike.

We will now consider spacetimes M which are neighbourhoods of spacelike tori T. The linear holonomy $q\rho : \mathbb{Z} \oplus \mathbb{Z} \to \mathbf{SO}(2,1)_0$ has abelian image. Therefore

(1) $q\rho$ has image in $\mathbf{SO}(2)$ up to conjugacy, or
(2) the image of $q\rho$ stabilizes a null vector, or
(3) the image is hyperbolic, i.e., conjugate into the subgroup of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\lambda & \sinh\lambda \\ 0 & \sinh\lambda & \cosh\lambda \end{pmatrix}$$

(in x, y, t coordinates.)

Proposition 7. *In cases 1) and 2) the linear holonomy is trivial, and a neighbourhood of T embeds in a complete spacetime.*

Proof. 1) The holonomy is abelian so lies either in $\mathbf{SO}(2)$ or in the subgroup of pure translations. By lemma 1, projection of $\mathsf{dev}\,\widetilde{T}$ on the plane $t = 0$ is bijective. The holonomy group acts on the plane $t = 0$ and projection intertwines these actions. Since the holonomy acts without fixed points on $\mathsf{dev}\,\widetilde{T}$, the linear holonomy must be trivial.

2) The linear holonomy is a homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{R} \subset \mathbf{SO}(2,1)_0$. If the generators are linearly independent over $\mathbb{Q}$, after an arbitrarily small change in the linear holonomy, they become dependent and then a basis for $\pi_1 T$ can be chosen so that $q\rho((0,1)) = \mathsf{id}$. Then if $A = \rho((0,1))$, $A\mathbf{x} = \mathbf{x} + \mathbf{n}$. If the linear holonomy of T is nontrivial, $\mathbf{n}$ must be a multiple of the null vector (say $(1,0,1)$) fixed by the linear holonomy. Let P be a point on $C = \mathsf{dev}\,\widetilde{T} \cap \{y = 0\}$. Then $A^n P = P + n(1,0,1)$ is on C. But the tangent to C is everywhere spacelike, so the mean value theorem is contradicted. □

Now suppose the holonomy is purely translational. Since $\mathsf{dev}\,\widetilde{T}$ is spacelike and projects onto $\mathbb{R}^2$, the holonomy must map the two generators of $\mathbb{Z} \oplus \mathbb{Z}$ onto independent spacelike vectors. Choose coordinates so these lie in $\mathbb{R}^2$. Then there is a compact spacetime with boundary which contains a neighbourhood of T in M and is the quotient of $C_1 \leq y \leq C_2$ by the holonomy group. Furthermore this spacetime can be extended to a complete spacetime.

Definition 2. *A spacetime containing a closed spacelike torus with linear holonomy nontrivial and lying in the group*

$$\{T(\lambda) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\lambda & \sinh\lambda \\ 0 & \sinh\lambda & \cosh\lambda \end{pmatrix}, \lambda \in \mathbb{R}\}$$

is called a standard spacetime *if it is of the form* $\{(x,y,t) : t^2 > x^2, t > 0\}/\langle A, B\rangle$ *where*

$$(*)\qquad \begin{aligned} A \cdot (x,y,t) &= T(\lambda) \cdot (x,y,t) + (0,e,0) \\ B \cdot (x,y,t) &= T(\mu) \cdot (x,y,t) + (0,f,0) \end{aligned}$$

and $(\lambda, e), (\mu, f)$ *are independent in* $\mathbb{R}^2$*.*

Proposition 8. *In case 3) a neighbourhood of* T *in* M *embeds in a standard spacetime (possibly after a change of time orientation).*

Proof. The holonomy is of the form (*) because $\langle A, B\rangle$ is abelian. The function $t^2 - x^2$ on $\widetilde{T}$ is well defined on T and therefore bounded on T. So we have $\mathsf{dev}\,\widetilde{T} \subset \{-C \le t^2 - x^2 \le C\} \times (-\infty < y < \infty)$. Also, $\mathsf{dev}\,\widetilde{T}$ projects onto the (x,y)-plane, but the projection of the region $t^2 - x^2 < 0$ is not connected. So $\mathsf{dev}\,\widetilde{T}$ must contain some points with $t^2 - x^2 \ge 0$. Since $\mathsf{dev}\,\widetilde{T}$ is spacelike, it does not lie entirely in $t^2 - x^2 = 0$. Suppose without loss of generality that $P \in \mathsf{dev}\,\widetilde{T}$ and $t(P) > 0$, $b^2 = t^2(P) - x^2(P) > 0$. Then we will show $t > 0$, $t^2(P) - x^2(P) > 0$ everywhere on $\mathsf{dev}\,\widetilde{T}$. Suppose $Q \in \mathsf{dev}\,\widetilde{T}$ and $(t^2 - x^2)(Q) = -c^2 < 0$. Then the orbit of Q lies in a half space say $x \le -c$. However $\mathsf{dev}\,\widetilde{T}$ projects 1-1 onto the surface $U = t^2(P) - x^2(P) = b^2$ and the projection is equivariant with respect to the action (*) of $\pi_1 T$ on U. So Q cannot exist. If there existed $R \in \mathsf{dev}\,\widetilde{T}$ with $(t^2 - x^2)(R) = 0$, then after an arbitrarily small isotopy, $\mathsf{dev}\,\widetilde{T}$ would be a spacelike surface on which $(t^2 - x^2)$ changed sign. So (up to a change of sign of t) $\mathsf{dev}\,\widetilde{T}$ lies in a region $t > 0$, $a^2 \le t^2 - x^2 \le b^2$, $-\infty < y < \infty$. The holonomy group acts properly discontinuously on that region. It follows that a neighbourhood of T embeds in a standard spacetime. $\square$

5. Domains of dependence and geodesic laminations

Definition 3. *Suppose* M *is a spacetime containing a closed spacelike future directed surface* S *of genus* $g > 1$*. The* domain of dependence *of* $\widetilde{S}$ *is the region* $D = D(\widetilde{S})$ *in* $\mathbb{R}^{2+1}$ *defined by* $x \in \widetilde{S}$ *or* x *is in the future of* $\widetilde{S}$ *or every future directed timelike or null ray through* x *meets* $\widetilde{S}$*. And similarly if* S *is past directed.*

Note that the spacetimes of proposition 2 and the standard spacetimes of definition 1 are domains of dependence. We remark that there is no loss in generality in assuming that $\widetilde{S}$ is future directed and M contains a closed smooth strictly convex surface S' in the future of S. The domains of dependence of $\widetilde{S'}$ and $\widetilde{S}$ are equal.

Proposition 9. $\pi_1 S$ *acts properly discontinuously on* $D(\widetilde{S})$*.*

Proof. Without loss of generality S is future directed. Given x in the past of $\widetilde{S}$, there is a neighbourhood U of x such that all future pointing timelike or null rays through x meet $\widetilde{S}$ in a compact set. The proper discontinuity of $\pi_1 S$ on $D(\widetilde{S})$ then follows from the proper discontinuity of the action on $\widetilde{S}$. $\square$

Proposition 10. *Suppose* M_1 *is a standard future directed spacetime and* M_2 *is a standard past directed spacetime with the same holonomy as* M_1*. Then the closures of the developments of* $\widetilde{M_1}$ *and* $\widetilde{M_2}$ *are disjoint.*

Proof. The intersection of the closures of the developments is a convex set. Because the restriction of t to the intersection is bounded above and below, the intersection is a bounded convex set. If it is nonempty, the holonomy must fix its barycenter. In this case, the holonomy is conjugate into $\mathbf{SO}(2,1)$. By definition, the development of a standard spacetime with holonomy in $\mathbf{SO}(2,1)$ must be a proper subset of the region L_+ or L_- of proposition 2. $\square$

The argument extends to the domains of dependence introduced below.

Definition 4. *Suppose M is a spacetime containing a closed spacelike surface S of genus $g > 1$. M is a* domain of dependence *if M is the quotient of $D(\widetilde{S})$ by $\pi_1 S$.*

Later we will extend the definition in a natural way to spacetimes of constant curvature. Recall that a null plane is one on which the Lorentzian structure is degenerate; equivalently it is a plane containing a null line but no timelike line. A null plane through the origin is the subspace orthogonal to the null line through the origin and contained in the given null plane. All null lines in a given null plane are parallel.

Proposition 11. *a) The boundary X of a domain $D = D(\widetilde{S})$ of dependence in $\mathbb{R}^{2+1}$ is convex. b) Any timelike linear function on $\mathbb{R}^{2+1}$ which increases into the future or the past according as $\widetilde{S}$ is future or past directed is bounded below on D. c) At any boundary point p, all supporting planes are null or spacelike. d) For each $p \in X$ there is at least one null supporting plane. Moreover, the cone on the set of normals to supporting planes is convex and the extreme rays of this cone are null rays. e) Null rays contained in X and containing p correspond one-to-one to null supporting planes at p.*

Proof. We assume $D = D(\widetilde{S})$ where $\widetilde{S}$ is future directed and convex. *b*) follows from proposition 10. Suppose q is not in the domain of dependence D. Then some future directed null ray l through q does not meet $\widetilde{S}$, because the union of the set of future directed timelike or null rays through p which meet $\widetilde{S}$ is convex set. Let P be the unique null plane through q containing l. Among all planes $P + (0,0,t)$ parallel to P and disjoint from D, the maximum t_0 is attained. So q lies in the union of the planes $P + (0,0,t), t \in (-\infty, t_0]$. So the domain of dependence is an intersection of open half spaces with boundaries which are null planes. This shows that D is convex and every point of X has a null supporting plane. In general, if a convex region in a vector space V is defined as the intersection of a set of half spaces $\{g(x) \geq C_g : g \in G\}$, the dual vectors to the supporting planes at a boundary point p form a compact convex set in (the projectivization of) V^* and the extreme points are elements of G which are minimized at p. Given a null ray l in X through $p \in X$, there is a supporting plane Q parallel to the null plane say P through l. Since D is future complete, Q cannot be a translate of P towards the past. Since $p \in X$, we must have $P = Q$. Now suppose $p \in X$ and P is a null supporting plane of D at p. Let l be the future directed null ray in P through p. By construction of the null supporting plane, l does not meet $\widetilde{S}$, so l is not in D. But D is future complete, so l is in the closure of D, so l lies in X. $\square$

Proposition 12. *Given a closed oriented hyperbolic surface S and a measured geodesic lamination on S, there is a corresponding flat Lorentz manifold M and an embedding of S in M such that M is a domain of dependence of S and the* *linear holonomy of S is the map $H : \pi_1 S \to \mathbf{SO}(2,1)_0$ which is the holonomy of S considered as a hyperbolic surface.*

Proof. [19] and [20] are recommended as introductions to geodesic laminations. First we consider the case where the measured geodesic lamination L is a finite union of simple closed curves. We start with a manifold $M_0 = L_+/\pi_1 S$ as in proposition 2. S lies in M_0 as the quotient of the hyperboloid $\widetilde{S}$ of unit timelike vectors. Geodesics on $\widetilde{S}$ are the intersection of planes through the origin with $\widetilde{S}$. M_0 contains a finite set of totally geodesic surfaces, each of which is the quotient of the positive light cone in a 2-dimensional Lorentzian spacetime by an infinite cyclic group which has two null eigendirections. Now construct a manifold with boundary, say M_1 by splitting M_0 along these totally geodesic surfaces. Suppose the curve $C_i \subset L$ has transverse measure a_i. Let $W_{i,+}$ and $W_{i,-}$ be the corresponding boundary components of M_1. We give $W_{i,+} \times [0, a_i]$ the product Lorentzian structure, so that the second factor is spacelike and orthogonal to the first factor and has length a_i. Then let $M = M_1 \cup \bigcup_i W_{i,+} \times [0, a_i]$ where $W_{i,+} \times \{a_i\}$ is identified with $W_{i,-}$. Then M is a flat Lorentz manifold with a closed spacelike convex surface (made from pieces of S and product annuli). There is a natural homotopy equivalence between M and M_0. The linear holonomy is unchanged. The element t of $H^1(\pi_1 S, \mathbb{R}^{2+1})$ which M represents may be described as follows: Orient C_i. Then $t = \sum_i a_i \cdot t_i$ where t_i is the extension by zero of the cohomology class (with support in an open annular neighbourhood $N(C_i)$) $\mathsf{PD}(C_i) \otimes \mathbf{e}_i$ where PD is Poincaré duality and $\mathbf{e}_i$ is the unit spacelike vector fixed by the holonomy of C_i. Each $\mathbf{e}_i$ is well defined in the system of local coefficients on S. We fix the orientation of $\mathbf{e}_i$ so that $\mathbf{u}, \mathbf{v}, \mathbf{e}_i$ is a positively oriented triple, where $\mathbf{u}, \mathbf{v}$ are the eigenvectors of the holonomy of C_i with corresponding eigenvalues less than, respectively greater than, 1. Now we consider the more general case of a measured lamination such that the support is not a finite union of closed leaves. $S - L$ is a union of finitely many regions A_i. Let the preimage of A_i in the universal cover $\widetilde{S}$ be the disjoint union of regions A_{ij}. Fix some region A_{00}. We will translate the cone over each remaining region to obtain the required spacetime. In case the support of the lamination is a finite union of closed geodesics, this construction will be the same as the preceding one, but we will look at it in the universal cover. For each geodesic $l \in \widetilde{L}$, let $\mathbf{e}_l$ be that unit vector orthogonal to the plane P_l which meets S in l which points away from A_{00}. Now given a path $h : [0,1] \to \widetilde{S}$ transverse to L, let $\mathbf{e} : [0,1] \to \mathbb{R}^{2+1}$ be a continuous function such that $\mathbf{e}(s) = \mathbf{e}_l$ if $h(s)$ is on the leaf l. Given a region A_{ij}, let h be a path with $h(0) \in A_{00}, h(1) \in A_{ij}$. Let $d\mu(s)$ denote the measure on $[0,1]$ determined by the transverse measure on $\widetilde{L}$ and the map h. Let

$$\mathbf{x}_{ij} = \mathbf{x}_h = \int_0^1 \mathbf{e}(s) d\mu(s). \tag{1}$$

Any other path with the same endpoints gives the same value, just as the ordinary transverse measure is not changed by homotopy with endpoints fixed.

Consider the union $\widetilde{S}'$ of the closures of the surfaces $A_{ij} + \mathbf{x}_{ij}$, which we will show to be disjoint. $\widetilde{S}'$ is a translate of $\widetilde{S}$ by a vector valued function which is discontinuous only across $\widetilde{L}_0$, the sublamination of $\widetilde{L}$ consisting of leaves of non-zero mass. Such leaves are the covers of closed isolated geodesics, and L contains no other closed geodesics. For each geodesic l in $\widetilde{S}$ which covers a closed geodesic in S, there correspond two geodesics l'_+ and l'_- which differ by a translation orthogonal to the planes in which they lie. Attach annuli as in the case of a lamination supported on finitely many simple closed curves to obtain a spacelike surface $\widetilde{S}''$. An alternative description will be useful to show that the surfaces $A_{ij} + \mathbf{x}_{ij}$ are disjoint. We define a function $\mathbf{x}$ on L_+. First let $\mathbf{x}$ be 0 in the open cone over A_{00}. Observe that the lamination $\widetilde{L}$ determines a transversely measured lamination of L_+, in which the leaves are the cones over the leaves of $\widetilde{L}$ (so they are totally geodesic Lorentzian surfaces). We will also use L to refer to this 2-dimensional lamination for simplicity.

Choose a basepoint q_0 in the cone over A_{00} and, for any p in L_+, choose a path h from q_0 to p transverse to $\widetilde{L}$. Define $\mathbf{x}(p) = \mathbf{x}_h$ where $\mathbf{x}_h$ is defined by (1). Then $f(p) = p + \mathbf{x}(p)$ defines a map which is continuous except across $\widetilde{L}_0$. By invariance of domain, to show that $f : L_+ - \widetilde{L}_0 \to f(L_+ - \widetilde{L}_0)$ is a homeomorphism it suffices to show that f is one-to-one. Since $\mathbf{x}$ is constant in complementary regions and on each leaf, in showing that $p + \mathbf{x}(p) = q + \mathbf{x}(q)$ implies $p = q$ we may assume that $p - q$ is null. We have a map $f : L_+ \to M$ defined by $f(p) = p + \mathbf{x}(p)$. Suppose $q \in L_+$ and $p - q$ is null and future directed.

Lemma 2. *Suppose P and Q are nondegenerate Lorentzian planes through the origin in $\mathbb{R}^{2+1}$. Suppose the line of intersection of P and Q is timelike or null. Then P and Q divide the hyperbolic plane (the hyperboloid of unit timelike future pointing vectors) into three regions E, F, G where the closure of E meets P but not Q, the closure of F meets P and Q, and the closure of G meets Q but not P. If $\mathbf{m}, \mathbf{n}$ are normals to P, Q respectively, oriented so $\mathbf{m}$ points into E and $\mathbf{n}$ points into F, then the scalar product $\langle \mathbf{m}, \mathbf{n} \rangle$ is positive.*

Proof. Suppose P is the plane orthogonal to $(0, 1, 0)$. If a plane Q had a normal $\mathbf{n}$ orthogonal to $(0, 1, 0)$, $\mathbf{n}$ would lie in P and being spacelike, would have a timelike orthogonal complement in P. Then P and Q would have timelike intersection. As any pair of planes satisfying the conditions of the lemma can be deformed through pairs of planes satisfying the conditions to any other pair, and the sign of the scalar product does not change we can assume that P and Q are almost parallel in which case the lemma is clear. □

By formula (1) applied to the ray from q to p, $\mathbf{x}(p) \cdot (p - q) \geq 0$. (Equality holds only if the line crosses no leaves of the lamination.) Furthermore, since the leaves of $\widetilde{L}$ are disjoint planes, lemma 2 implies that $\langle \mathbf{x}(q + t(p - q)), \mathbf{x}(q + t(p - q)) \rangle$ increases as t increases (and the increase is strict unless the path does not meet $\widetilde{L}$). So $\mathbf{x}(p)$ is spacelike or equal to 0. Now $f(p) - q$ is null or spacelike:

$$\langle f(p) - q, f(p) - q \rangle = \langle p - q + \mathbf{x}(p), p - q + \mathbf{x}(p) \rangle = 2\langle p - q, \mathbf{x}(p) \rangle + \langle \mathbf{x}(p), \mathbf{x}(p) \rangle \geq 0.$$

Equality holds only if $f(p) = p$. This establishes that if $f(p) = f(q)$ and $q \in A_{00}$ then $p = q$. But if the basepoint is chosen anywhere else in L_+, f is changed only by a constant translation, so f is injective in L_+ and, if L has no isolated leaves, $f(L_+)$ is the universal cover of our new spacetime. If L has closed leaves, product regions must be added to $f(L_+)$. We now wish to show that $\widetilde{S}''$ is equivariant with respect to a homomorphism $\rho : \pi_1 S \to \mathbf{ISO}(2, 1)$ such that the linear holonomy $q\rho : \pi_1 S \to \mathbf{SO}(2, 1)$ is the identity map to $\pi_1 S$. We will construct a cocycle t. Choose a lift $\widetilde{p}$ of a basepoint p in A_{00}. Given $\alpha, \beta \in \pi_1 S$, choose paths $h_\alpha, h_\beta : [0, 1] \to S$ with $h_\alpha(0) = h_\alpha(1) = h_\beta(0) = h_\beta(1) = p$ which represent α, β and are transverse to L. Let $\widetilde{h_\alpha}$ be the lift of h_α beginning at $\widetilde{p}$ and ending at q say. Let $\widetilde{h_\beta}$ be the lift of h_β beginning at q and ending at r say. Define $t_\alpha = \mathbf{x}_{\widetilde{h_\alpha}}$, and similarly for any element of $\pi_1 S$. Representing $\alpha\beta$ by the path $\widetilde{h_\alpha}$ followed by $A \cdot \widetilde{h_\beta}$ we have (splitting the integral into two terms)

$$t_{\alpha\beta} = t_\alpha + A \cdot t_\beta.$$

Evidently $\widetilde{S}'$ is equivariant with respect to the action $\alpha x = A \cdot x + t_\alpha$. The interior of the union of the cones over the closures of the surfaces $A_{ij} + \mathbf{x}_h$ (where h is a path from A_{00} to A_{ij}) gives a spacetime $\widetilde{M}$ unless L contains closed leaves. In this case, we add in regions congruent to $K_+ \times [0, a_i]$ where K_+ is a positive light cone in dimension $1+1$, and a_i is the transverse measure of a closed leaf. $\pi_1 S$ acts and the quotient spacetime M evidently is future complete and contains a closed convex spacelike surface $S'' = \widetilde{S}''/\pi_1 S$ which will be strictly convex if there are no closed leaves. Finally, let us show that $\widetilde{M}$ and therefore also M is a domain of dependence as claimed. Intuitively, $\widetilde{S}$ stretches and so at least as many timelike and null rays hit $\widetilde{S}''$ as hit $\widetilde{S}$.

Suppose that L has no isolated leaves so that f is a homeomorphism. Suppose $q \in L_+$, $p - q$ is null and future directed, and p is on $\widetilde{S}$. Recall that $\widetilde{S}$ is convex. Let the null and timelike rays from q meet $\widetilde{S}$ in the disc D with boundary C corresponding to the null rays. Let R be a spacelike plane in the future of q and let D' and $f(D)'$ be the radial projections from q of D and $f(D)'$ on R. Let $f(C)'$ be the projection of $f(C)$. Then from $\langle f(p) - q, p - q\rangle = \langle \mathbf{x}(p), p - q\rangle \geq 0$ with equality only if $\mathbf{x}(p) = 0$, and $f(p) = p$ it follows that $f(C)'$ has winding number 1 around any point in the interior of D'. So $f(D)' \supseteq D'$. So q is in the domain of dependence of $f(\widetilde{S}) = \widetilde{S}''$. If L contains isolated leaves, the transverse measure may be smoothed out over a neighbourhood of each isolated leaf and the resulting function say f' is a homeomorphism and the foregoing argument applies. We may use any A_{ij} as a reference domain in which the function $\mathbf{x} = 0$, for some function f'' (which differs from f by a constant translation; $f''(r) = f(r) - \mathbf{x}(s)$ (for some $s \in A_{ij}$)) so for every point q in L_+ which is in the closure of some component of $L_+ - L$, $f(q) = q + \mathbf{x}(q)$ is in the domain of dependence of $\widetilde{S}''$. Not every point in L_+ lies in the closure of a complementary component. Let q lie on a leaf which is not a boundary leaf. Then f is a homeomorphism in a neighbourhood of q so $f(q)$ is in the closure of the domain of dependence of $\widetilde{S}''$. Since f is open, $f(Q)$ must lie in the domain of dependence of $\widetilde{S}''$. So $\widetilde{M}$ is contained in the domain of dependence of $\widetilde{S}'$. By construction, through each point in the frontier of $\widetilde{M}$ there is a null ray which does not meet $\widetilde{S}$. So M is a domain of dependence. □

Proposition 13. *Given a spacetime M which is the domain of dependence of a closed spacelike surface S of genus $g > 1$ the geometry of the frontier of the universal cover determines a measured geodesic lamination on S in the hyperbolic structure on S determined by its linear holonomy, and this correspondence is inverse to that of Proposition 12.*

Proof. Let X denote the frontier of $D(\tilde{S})$. Given a point r on X, let $N(r)$ be the union of the future pointing null rays in X based at r. It is a closed subset of X. Let $F(r)$ be the convex hull of $N(r)$.

Claim. Suppose $r \neq s$. Then $F(r)$ and $F(s)$ are disjoint unless r and s lie on a single null ray. Proof of claim: Choose a timelike direction and let p project X orthogonally onto an orthogonal spacelike plane which we identify with $\mathbb{R}^2$. Rays project to rays. If there is a unique null ray l through r say, then $F(r) = l$. If $t \in F(r) \cap F(s)$ and $t \neq r$ then if t was timelike from s the future of t on l would be timelike from s. But that would contradict $l \subset X$. So $t \in N(s)$. Since there is a unique null line in X through t, we must have that r and s lie on a single null ray. So we assume that each of $N(r)$, $N(s)$ consists of at least two null rays. There is a pair of extreme rays, say $p(l_1), p(l_2)$ through $p(r)$ such that one component of $\mathbb{R}^2 - l_1 \cup l_2$ contains $p(s)$ and is disjoint from $p(N(r))$. We may choose the direction of projection so that $p(r) = 0$ and $p(l_1) \cup p(l_2)$ span the plane $x = 0$. Let l_1', l_2' be the extreme rays through s_1 so $p(r)$ lies in the component of $\mathbb{R}^2 - (p(l_1') \cup p(l_2'))$ which is disjoint from $N(s)$. Since $p(l_1'), p(l_2')$ don't cross $p(l_1), p(l_2)$, the x-components of the tangent vectors to $p(l_1'), p(l_2')$ must be nonnegative. It follows that the plane sector spanned by l_1, l_2 does not meet the plane $x = 0$. So (for some small $c > 0$) the plane $x = c$ separates $F(r)$ from $F(s)$. This proves the claim.

Let $D = D(\widetilde{S})$. For each $y \in D$ the past of y in $\mathbb{R}^{2+1}$ is foliated by strictly convex hyperboloids of constant separation from y, so there is a unique point $a(y)$ on X where separation from y is maximized, and $a(y)$ depends continuously on y. The tangent plane to the hyperboloid $\{p : \langle p - y, p - y\rangle = \langle a(y) - y, a(y) - y\rangle\}$ is a spacelike support plane at $a(y)$, and y is on the timelike normal through $a(y)$ of this spacelike plane. So every point in D is in some set $F(r)$. Let E be the subset of points in X with at least 3 supporting null planes, let F be the subset of points with exactly 2 supporting null planes, and G the subset of points through which pass exactly one null ray. Define a lamination L^* of $\widetilde{M}$ by $L^* = D - \cup_{r\in E} \mathsf{int} F(r)$. Then L^* is the disjoint union of $F(r)$ $(r \in F)$ and the faces of the cones $F(r)$ $(r \in E)$ and L^* is closed, so L^* is a lamination. By construction, L^* is preserved by the action of the deck group $\pi_1 S$, and so it projects to a lamination on the surface $T = \{y : y \in D, \langle y - a(y), y - a(y)\rangle = -1\}$. T is convex because the future of T is the union of the convex regions $U(p) = \{y \in D : \langle y - p, y - p\rangle > -1\}$ where $p \in X$. Each point y on T has a unique tangent plane, because the support plane at $a(y)$ is parallel to the tangent plane at y. A convex surface has a continuously varying tangent plane if and only if every point has a unique support plane, so we have a well defined and continuous map $t : T \to \mathbb{H}^2$ which takes each point to the point on the hyperboloid of unit timelike vectors with a parallel tangent plane. Note that t extends naturally to $t : D \to \mathbb{R}^{2+1}$; map the ray from $a(p)$ through p isometrically to the ray from 0 through $t(p)$. We will define a transverse measure on L^* so that with respect to the induced transverse measure on $L = t(L^*)$, t is the inverse of the map f defined in proposition 13. We note that *i)* $t(L)$ is a union of totally geodesic subspaces *ii)* t is onto because every timelike unit vector is the normal to some support plane of D *iii)* t is isometric on each component of $\cup_{r\in E} \mathsf{int}(F(r))$ and so $t(D - L)$ is open. It follows that $t(L)$ is closed, so $t(L)$ is a geodesic lamination. Define a vector valued function $\mathbf{x} : \mathbb{H}^2 \to \mathbb{R}^{2+1}$ by $t(p) - p = \mathbf{x}(p)$. Then $d\mathbf{x}$ is a vector valued measure supported on L and (because t takes leaves of L^* to leaves of L) orthogonal to L. So $d\mathbf{x}$ can be identified with a transverse measure to the geodesic lamination L, and this measure is positive (orienting L, as in proposition 12, so that the normal always points away from some fixed region) because the regions $t^{-1}(F(r))$ $(r \in E)$ are disjoint. This establishes proposition 13. $\square$

We now have, for each hyperbolic structure on S, a bijection from the space $\mathsf{ML}(S)$ of measured geodesic laminations on S to $H^1(\pi_1 S, \mathbb{R}^{2+1})$. It is clear that the cocycle constructed in proposition 12 depends continuously on the measured geodesic lamination, and that the map $[\mathbf{x}] : \mathsf{ML}(S) \times \mathsf{Teich}(S) \to \mathbf{R}^{6g-6}$ defined by any continuous trivialization of the bundle $H^1(\pi_1 S, \mathbb{R}^{2+1})$ is continuous. It follows that the space $\mathsf{ML}(S)$ of measured geodesic laminations is homeomorphic to $\mathbb{R}^{6g-6}$. This is analogous to proving the same result by showing that bending laminations parametrize the space of quasifuchsian deformations of a Fuchsian group holding the hyperbolic structure fixed on one boundary component of the convex hull as in [14, 20]. As $\mathbf{SO}(2,1)$ modules, $\mathbb{R}^{2+1}$ and the Lie algebra $\mathfrak{so}(2,1)$ are canonically isomorphic (and so also canonically dual). Witten's paper [2] contains the result that the Einstein equation $R_{ij} = 0$ can be regarded as the Euler-Lagrange equations for the Chern-Simons form $\langle A, dA + \frac{2}{3}[A, A]\rangle$ where A is an $\mathbf{ISO}(2,1)$ connection and $\langle\ ,\ \rangle$ is the invariant bilinear form on the Lie algebra $\mathfrak{iso}(2,1)$ for which both the radical $\mathbb{R}^{2+1}$ and any Levi factor $\mathfrak{so}(2,1)$ are isotropic; this form is unique up to multiplication by a constant factor and defines a dual pairing between $\mathfrak{so}(2,1)$ and $\mathbb{R}^{2+1}$. We have shown that (future directed) domains of dependence form a family parametrized by the bundle of cohomology spaces $V = H^1(\pi_1 S, \mathbb{R}^{2+1}) \to \mathsf{Teich}(S)$, where $\pi_1 S$ varies in Teichmüller space. Using the dual pairing and Poincaré duality, this identifies the bundle V with the cotangent bundle to Teichmüller space. By a theorem of Goldman [21], the pairing determines a symplectic structure on $T^*\mathsf{Teich}(S)$, and this is the standard symplectic structure on a cotangent bundle (rather than the symplectic structure induced from the symplectic structure of Teichmüller space). Of course we can also identify the bundle V with the tangent bundle to Teichmüller space, because the inner product on $\mathfrak{so}(2,1)$ determines a symplectic structure on Teichmüller space and so an identification between the tangent bundle and the cotangent bundle.

We now wish to give a more detailed description of the geometry of the boundary X. First we recall that a geodesic lamination on a hyperbolic surface has measure 0. It follows that the map $a \circ t^{-1} : \mathbb{H}^2 \to X$ maps almost all tangent planes to the countable set C.

Proposition 14. *Suppose X contains two parallel null rays l_1 and l_2. Then X contains two null rays l_0 and l_3 parallel to l_1 and l_2 with basepoints r and q such that the seqment rq is spacelike and lies in X and from each point in rq there is a unique null ray in X parallel to l_1. Moreover every null ray in X parallel to l_1 has basepoint in rq, and there are parallel null rays l_4 and l_5 through r and q such that through every point on rq there is a null ray in X parallel to l_4 and all null rays in X parallel to l_4 have basepoint in rq. $\pi_1 S$ contains an infinite cyclic subgroup $\langle A\rangle$ which is the stabilizer of rq and preserves each null ray parallel to l_1 or to l_2. $\langle A\rangle$ is the fundamental group of an isolated leaf of the geodesic lamination determined by X. r, q are in C.*

Proof. Suppose l_1 is a null ray based at r. Then $F(r) \cap T$ contains a unique geodesic g_1 asymptotic to l_1. $t(g_1)$ is a geodesic in the lamination $L = t(L^*)$.

Lemma 3. *The support of the transverse measure on L is all of L.*

Proof. Otherwise there would be a closed leaf C_1 or a spiralling leaf C_2 with transverse measure 0. But then $t^{-1}C_i$ would be a single geodesic. Since the cone on $t^{-1}C_i$ separates two complementary components $F(r_1)$ and $F(r_2)$ we must have $r_1 = r_2$. But then $t^{-1}C_i$ is not part of L^* so C_i is not part of L. $\square$

Lemma 4. *If L is a measured geodesic lamination on a closed surface for which L is the full support of the transverse measure, if m_1, m_2 are distinct geodesics which are asymptotic to each other, then m_1 and m_2 are both boundary leaves of a complementary region A_i, and so each of m_1, m_2 has transverse measure* 0.

Proof. The proof of lemma 4.5 of [19] establishes this. $\square$

Consider all the null rays $\{l_i\}_{i \in I}$ parallel to l_1 in X. For each such null ray l_i, we have $l_i \subset N(r_i)$ for a unique $r_i \notin G$. $F(r_i) \cap T$ contains a unique geodesic m_i asymptotic to l_i. $t(g_i)$ is independent of i, because if say $m_1 \neq m_2$ then for all $i \in I$, $t(l_i) = m_1$ or m_2 (by lemma 4.) Since $t \circ f = \mathsf{id}$, this contradicts the fact that each m_i has transverse measure 0.

Let B be the set of basepoints of the rays parallel to l_1. Then $D \cap \cup_{b \in B} F(b) = t^{-1}(C)$ for some isolated leaf C, and so B is a spacelike line segment of length $\mu(C)$ where μ is the transverse measure. Let r, q be the end points of B and l_4, l_5 the null rays parallel to l_1 through r, q. r, q are in E (i.e., they have at least 3 supporting null planes) because $\mathsf{int}(F(r))$ and $\mathsf{int}(F(s))$ are distinct complementary regions. $\pi_1 C$ is the infinite cyclic subgroup $\langle A \rangle$ of the statement of the proposition. $\square$

We call the parts of X which are ruled by parallel null rays the *ruled regions* and their union the *ruled part.*

Proposition 15. *Given a spacetime $M = D(\widetilde{S})$, there does not exist a spacetime M' strictly containing M such that the development of $\widetilde{M}'$ does not meet the ruled part of X. In particular, if the holonomy of M fixes a point or the corresponding geodesic lamination has no isolated leaves, M is a maximal spacetime.*

Proof. We use the notation D, X as in propositions 13 and 14. Suppose M' is a spacetime strictly containing M, with development $d : \widetilde{M}' \to \mathbb{R}^{2+1}$ and let $\widetilde{M}''$ be the component of $d^{-1}(D \cup X)$ which contains $\widetilde{M} \subset \widetilde{M}'$. First we consider the case that M has holonomy conjugate into $\mathbf{SO}(2,1)_0$. We may assume $M = L_+/\pi_1 S$.

If $0 \in \widetilde{M}''$, then given $g \neq 1$ and a geodesic $C = [x, g\cdot x]/\{g\cdot x = x\}$ in the surface $x \cdot x = -1$, the cone on C lies in $\widetilde{M}''$ so C is null homotopic. But C has nontrivial holonomy g. Now suppose $0 \neq x \in \widetilde{M}'' \cap X$. There is a neighbourhood U of x in $X \cap \widetilde{M}''$ such that U is homeomorphic to an open disc. Since $\pi_1 S$ does not act properly discontinuously on $\mathsf{cl}\, L_+$, there exist $q, g\cdot q \in U$ and a path $[q, g\cdot q]$ in U. Moreover, choosing x on the surface T, there is a homotopy from $[x, g\cdot x]$ to $[q, g\cdot q]$ which moves each point along a straight line. But then the loop $[x, g\cdot x]/\{x = g\cdot x\}$ is homotopic into U and we have a contradiction as before.

Now suppose that the holonomy does not fix a point. By a "parallel region" we mean a subset of X which is maximal with respect to being a union of parallel null rays. The union of the parallel regions is a closed subset of X, because it is the intersection with X of the closure in $D \cup X$ of the preimage by t of the union of the isolated leaves of L, and t is continuous. There is no open subset of X where the direction of the null rays is nonconstant and varies continuously. For if there was, there would be two null rays l_1 and l_2 and a sector A of the circle S^1 of null directions bounded by the directions of l_1 and l_2, such that the null ray with direction in A is unique and there is a continuous map $b : A \times [0, \infty] \to X$ such that $b(\{a\} \times [0, \infty])$ is a null ray in the direction a. But $\pi_1 S \cdot A = S^1$ so outside a compact region, X is foliated by null rays. Then the holonomy fixes the barycenter of the convex hull of the set of bases of the null rays in X. Given $x \in U$ where U is disjoint from the ruled regions, we will find points $q \in E$ arbitrarily near x. If there was a neighbourhood of x in U consisting of points with unique supporting plane, then x would have a neighbourhood foliated by null rays. So there are points arbitrarily near x in E or F. If $\mathbf{n}$ is a timelike direction and the supporting plane $P(\mathbf{n})$ with normal $\mathbf{n}$ meets X in a single point (which is the case unless $P(\mathbf{n})$ meets X in a line segment which is the base of a parallel region, using proposition 14) then the map $\mathbf{m} \mapsto P(\mathbf{m})$ is continuous at $\mathbf{n}$. Given a point r in F and a timelike normal $\mathbf{m}$ to a supporting plane at r, there is a timelike normal $\mathbf{u}$ arbitrarily close to $\mathbf{m}$ for which $P(\mathbf{u})$ is a supporting plane at some point of E, because the set of timelike directions normal to supporting planes at E corresponds (using t) to an open dense subset of the hyperbolic plane, namely $\mathbb{H}^2 - \widetilde{L}$. So taking $q = P(\mathbf{m}) \cap X$, we have seen that E is dense in the complement of the ruled regions of X.

Suppose the holonomy does not fix a point and $x \in U \subset \widetilde{M}'' \cap X$ and U is a simply connected open set disjoint from the ruled regions. Then there exist infinitely many points of E in U. Given $r \in E$, $F(r)$ is one of the complementary components of the lamination L^*. Since there are only finitely many $\pi_1 S$ orbits of complementary components, there are points q and $g \cdot q$ in U with $q \in E, g \neq 1$. Choose $x \in D$ and paths $[x, g \cdot x]$ and $[q, g \cdot q]$ and join them by a straight line homotopy $H : [0,1] \times [0,1] \to D \cup X$ such that $H : \{s\} \times [0,1] \to D \cup X$ is the affine parametrization of the straight line joining $H(s,0) \in [x, g \cdot x]$ to $H(s,1) \in [q, g \cdot q]$. This projects to a homotopy which moves the loop $[x, g \cdot x]/\{x = g \cdot x\}$ into the simply connected neighbourhood U. This is a contradiction. □

The need for care is illustrated by Figure 1 which shows the fundamental domain of flat Lorentz manifolds with infinite cyclic fundamental group in dimension $1+1$, the development of which contain the origin or which contain closed timelike curves.

These examples can be "inserted" into the ruled part of the boundary of a domain of dependence, yielding manifolds that contain closed timelike curves.

In Lorentzian manifolds, spacelike surfaces of constant mean curvature solve the variational problem of maximizing surface area while enclosing a given volume. Because spacelike surfaces are locally graphs with bounded slope, it seems one should be able to deduce the regularity of maximizers. So one would expect each of the domains of dependence to have a foliation by surfaces S_t of constant mean curvature such that t is the volume of the past of the surface S_t, but we leave both existence and uniqueness as questions. Moncrief [55] discusses foliations by surfaces of constant mean curvature from a rather different point of view. One might also ask for a foliation by surfaces of constant curvature as in [52].

6. The case of de Sitter space

$\mathbb{RP}^3$ is separated by the quadric $Q : X^2 + Y^2 + Z^2 = W^2$ into two regions. The first, $X^2 + Y^2 + Z^2 < W^2$ is the Klein model of hyperbolic space $\mathbb{H}^3$. The region $X^2 + Y^2 + Z^2 > W^2$ is the union of the plane $W = 0$ (which can be regarded as the projective plane at infinity associated with $\mathbb{R}^3$) and the unbounded component

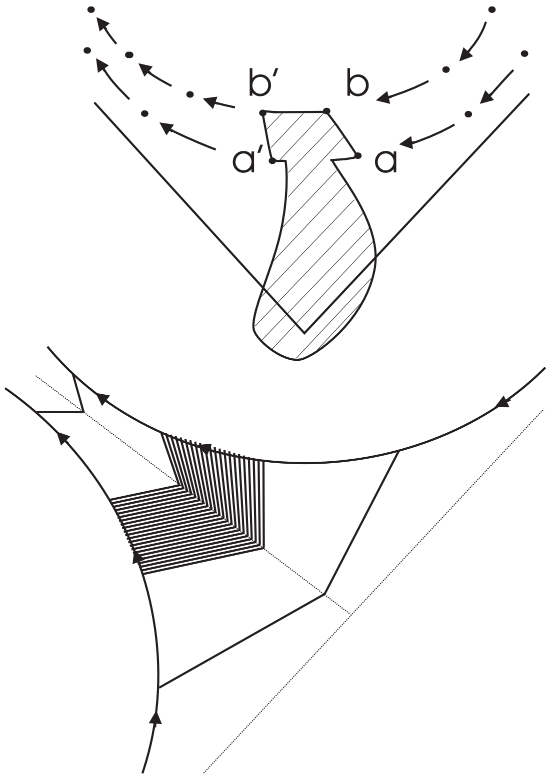

Figure 1. The first example has a fundamental domain which contains the fixed point of the holonomy. The development of the universal cover is not an embedding. In the second example part of the manifold is foliated by closed timelike curves.

of the complement of the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$, identifying $\mathbb{R}^3$ with the open set $W \neq 0$ in $\mathbb{RP}^3$ and setting $x = X/W, y = Y/W, z = Z/W$. This region $X^2 + Y^2 + Z^2 > W^2$ is de Sitter space. Topologically it is a twisted $\mathbb{R}$-bundle over $\mathbb{RP}^2$. It carries a Lorentzian metric invariant under the group $\mathbf{O}(3,1)/\pm 1 \cong \mathbf{O}_\uparrow(3,1)$ of projective transformations which map the quadric Q to itself. As a homogeneous space, de Sitter space is $\mathbf{O}_\uparrow(3,1)/\mathbf{O}_\uparrow(2,1) = \mathbf{SO}(3,1)_0/\{\pm 1\} \times \mathbf{SO}(2,1)_0$ as the stabilizer of $(0:0:0:1)$ is $\mathbf{O}_\uparrow(2,1)$. Of course $\mathbf{SO}(3,1)_0 \cong \mathbf{PSL}_2\mathbb{C}$. The double cover of de Sitter space is also called de Sitter space; it may be realized by the hyperboloid model, that is, as the set of unit spacelike vectors in $\mathbb{R}^{3+1}$, i.e., the vectors $\mathbf{x}$ such that $\mathbf{x} \cdot \mathbf{x} = 1$. De Sitter space has constant curvature $+1$. Thus spacelike geodesics focus, and, in the simply connected de Sitter space, there is a totally geodesic sphere of constant curvature 1 tangent to any spacelike plane in the tangent space of any given point. In the hyperboloid model these are the intersections of spacelike planes through the origin with the hyperboloid of unit spacelike vectors.

We will prefer to work in the Klein model, i.e., in $\mathbb{RP}^3$. Although it is not orthochronous, the Klein model of de Sitter space, like the Klein model of hyperbolic space, has the advantage that geodesics are straight lines. A geodesic is timelike, null or spacelike according as it meets the quadric Q in $2, 1,$ or 0 points. Given a point P in de Sitter space, draw the tangent cone from P to the quadric Q; this is the null cone of P. The planes in $\mathbb{RP}^3$ which do not meet Q are the totally geodesic spacelike planes.

Recall that a *projective structure* on a 2-manifold F is a $(\mathbf{PSL}_2\mathbb{C}, \mathbb{CP}^1)$-structure on F, or in other words an atlas of charts $\phi_\alpha : U_\alpha \to \mathbb{CP}^1$ where $U_\alpha \subset F$ is open and connected such that $\phi_\beta \circ \phi_\alpha^{-1}$ is the restriction of a Möbius transformation. F has a unique structure of a complex manifold such that the charts ϕ_α are holomorphic. Classically, the set $P(F)$ of projective structures up to isotopy on a closed surface F is described as a holomorphic bundle of affine spaces over the Teichmüller space $\mathsf{Teich}(F)$. Any affine space has an associated vector space of translations, and the vector bundle over Teichmüller space thereby associated with $P(F)$ is the cotangent bundle of Teichmüller space. The fiber $\pi^{-1}p$ over $p \in \mathsf{Teich}(F)$ consists of all those projective structures with the same underlying holomorphic structure (up to isotopy). Given any two projective structures $A, B \in \pi^{-1}p$, the difference $A - B$ is defined and is an element of the cotangent space at $p \in \mathsf{Teich}(F)$. Explicitly, the Schwarzian derivative of one projective structure with respect to the other defines a holomorphic quadratic differential on F.

Because Teichmüller space is a Stein manifold, there must be a global holomorphic section of $\pi : P(F) \to \mathsf{Teich}(F)$. The Fuchsian family of projective structures, $f : \mathsf{Teich}(F) \to P(F)$ defined by holomorphically mapping the universal cover of F to the unit disc, so that the monodromy is a Fuchsian group, is real but not complex analytic. The Bers embedding gives a holomorphic section. That is, fix $q \in \mathsf{Teich}(F)$. For each $p \in \mathsf{Teich}(F)$ there is a quasifuchsian group $i_p(\pi_1 F) \subset \mathbf{PSL}_2\mathbb{C}$ with the following properties. The homomorphism i_p is faithful with discrete image, and the limit set L_p of $i_p(\pi_1 F)$ is homeomorphic to a circle. L_p divides $\mathbb{CP}^1$ into two regions U_+ and U_-. $U_+/i_p(\pi_1 F)$ and $U_-/i_p(\pi_1 F)$ are Riemann surfaces corresponding to p, q. $U_+/i_p(\pi_1 F)$ has a natural projective structure and this defines a holomorphic section $i : \mathsf{Teich}(F) \to P(F)$. The holonomy map $\rho : P(F) \to \mathsf{Hom}(\pi_1 F, \mathbf{PSL}_2\mathbb{C})/\mathbf{PSL}_2\mathbb{C}$ (where the right action of $\mathbf{PSL}_2\mathbb{C}$ is by conjugation) is a holomorphic submersion.

There is an alternative description of $P(F)$ due to Kulkarni [34] and Thurston (communicated through [35]). This is related to work of Apanasov [56]. Suppose $d : \widetilde{F} \to \mathbb{CP}^1$ is the development map of a projective structure. Over $\mathbb{CP}^1$ there is an $\mathbb{R}$-bundle $H(\mathbb{CP}^1) = \mathbf{PSL}_2\mathbb{C}/(\mathbf{SO}(2) \ltimes \mathbb{C})$ where $\mathbf{SO}(2) \ltimes \mathbb{C}$ is a subgroup of the stabilizer $\mathbb{C}^* \ltimes \mathbb{C}$ of $\infty \in \mathbb{CP}^1$. The fiber over $z \in \mathbb{CP}^1$ can be thought of as the family of horospheres in hyperbolic space tangent to $\mathbb{CP}^1$ at z. There is also the bundle $D(\mathbb{CP}^1) = \mathbf{SO}(3,1)_0/\{\pm 1\} \times \mathbf{SO}(2)$ of round discs with distinguished point in the interior of the disc: The fiber over z is the set of round discs containing z in their interiors. Now given $p \in \widetilde{F}$, let K be a subset of $\widetilde{F}$ containing p such that $d|_K$ is an embedding and the closure of $d(K)$ is a round disc and K is maximal with respect to this property. Call K a *maximal disc* for p. A maximal disc K determines a totally geodesic hyperplane in hyperbolic space which meets the sphere at infinity in the boundary circle of the closure of $d(K)$. Given a maximal disc for p, consider larger and larger horospheres based at p. Eventually there is a horosphere tangent to the geodesic hyperplane associated with the maximal disc. So a maximal disc has a height, which is an element of the fiber of $H(\widetilde{F})$ over $\widetilde{F}$; the height is the natural projection $D(\mathbb{CP}^1) \to H(\mathbb{CP}^1)$. Of the maximal discs for p, there is one which is furthest away from p, in the sense that the height above p is maximal, and this is unique. Indeed, if K_1 and K_2 are maximal discs for p, which without loss of generality can be taken to be the point $\infty \in \mathbb{CP}^1$, then $d(K_1) - \{p\}, d(K_2) - \{p\}$ are the exteriors of two circles $\partial d(K_1), \partial d(K_2)$ in the Euclidean plane. If $\partial d(K_1)$ and $\partial d(K_2)$ are disjoint or meet in a single point, then $\widetilde{F}$ contains a region which maps conformally to all of $\mathbb{CP}^1$ or to all but one point of $\mathbb{C}$, which is absurd. If $\partial d(K_1)$ and $\partial d(K_2)$ intersect in points A, B, the outside of the circle with diameter AB has preimage K_3 and the geodesic hyperplane associated with K_3 is higher above p than the corresponding hyperplanes for K_1 or K_2. (In the upper half space model with p at ∞, the hyperplanes are hemispheres or (exceptionally) vertical half planes and the higher a hyperplane is above ∞ the lower its tangent horosphere, which is a horizontal plane, is). The height defines a section $h : \widetilde{F} \to H(\widetilde{F})$ where $H(\widetilde{F})$ is the pullback by $d : \widetilde{F} \to \mathbb{CP}^1$ of the bundle $H(\mathbb{CP}^1) \to \mathbb{CP}^1$. The highest maximal disc depends continuously on $p \in \widetilde{F}$. Define a map $D : \widetilde{F} \times [0, \infty) \to \mathbb{H}^3$ as follows. Given $p \in \widetilde{F}$, let $K(p)$ be the highest maximal disc at p and let $x(p)$ be the point of tangency of $K(p)$ with the horosphere based at p. Then D maps $\{p\} \times [0, \infty)$ isometrically to the geodesic from $x(p)$ to $d(p) \in \mathbb{CP}^1$, and extends to $D : \widetilde{F} \times [0, \infty] \to \mathbb{H}^3 \cup \mathbb{CP}^1$ such that $D|_{\widetilde{F} \times \{\infty\}} = d : \widetilde{F} \to \mathbb{CP}^1$. $\widetilde{F}$ may contain two overlapping maximal discs K_1 and K_2, and then $d(K_1 \cap K_2)$ is a crescent shaped part of the sphere, and $K_1 \cup K_2$ is foliated by curves each of which is mapped by d to a circular arc on $\mathbb{CP}^1$, and these circular arcs all have the same pair of endpoints. So $\widetilde{F}$ contains (at most countably many, but possibly 0) closed regions B_i, each topologically a product $\mathbb{R} \times [0, \theta]$ such that each $\mathsf{cl}\,(d(\mathbb{R} \times \{t\}))$ is a circular arc joining p_i, q_i, and that $d(\mathbb{R} \times \{t_1\})$ and $d(\mathbb{R} \times \{t_2\})$ intersect at an angle of $|t_1 - t_2|$ if $|t_1 - t_2| < \pi$. Note that θ may be bigger than 2π. The product structure on B_i can be chosen so that each $d|_{\{t\} \times [0,\theta]}$ is a submersion with image a circular arc orthogonal to each $d(\mathbb{R} \times \{s\})$.

Now we consider the Lorentzian dual of the Kulkarni-Thurston construction. Given $p \in \widetilde{F}$ consider the geodesic through p orthogonal to $K(p)$. Define $q : (0, \infty) \times \widetilde{F} \to \mathbf{SO}(3,1)_0/\{\pm 1\} \times \mathbf{SO}(2,1)_0$ by mapping (t, p) to the round disc which bounds a geodesic hyperplane at distance t from $K(p)$ along the geodesic through p orthogonal to $K(p)$. Thus this map $q : (0, \infty) \times \widetilde{F} \to \mathbf{SO}(3,1)_0/\{\pm 1\} \times \mathbf{SO}(2,1)_0$ factors through $\mathbf{SO}(3,1)_0/\{\pm 1\} \times \mathbf{SO}(2)$. The map q is locally injective and so defines a structure of a de Sitter manifold, i.e. a $(\mathbf{SO}(3,1)_0, \mathbf{SO}(3,1)_0/\{\pm 1\} \times \mathbf{SO}(2,1)_0)$-structure on $(0, \infty) \times \widetilde{F}$. By construction q is equivariant with respect to the holonomy map.

Proposition 16. *For any genus g, there is a family of manifolds, $\pi : X \to P(F)$, such that each fiber $\pi^{-1}\{a\}$ is a de Sitter manifold which is a future complete maximal domain of dependence, foliated by strictly locally convex spacelike hypersurfaces. Each admits a conformal compactification at future infinity, and the projection π to $P(F)$ is given by assigning to a given spacetime the projective structure on the Riemann surface at future infinity. (There is a second family of past complete spacetimes which differs only by the covering transformation of the covering of the non simply connected de Sitter space by simply connected de Sitter space.)*

Proof. For a given projective structure, q defines the structure of a de Sitter manifold on $(0, \infty) \times \widetilde{F}$, and taking the quotient by $\pi_1 F$ represented in $\mathbf{PSL}_2\mathbb{C}$ by the holonomy of the projective structure gives the required foliated spacetime. For any positive t, the surface $\{t\} \times \widetilde{F}$ is the dual in de Sitter space of the surface in $\mathbb{H}^3$ at distance t from the locally convex bent surface $x(\widetilde{F})$ in the direction toward infinity. So it is the dual of a locally strictly convex surface which has t as its smaller principal curvature at each point. (Given a surface in hyperbolic space, the dual surface consists of the points dual to the tangent planes to the surface.) The dual of a strictly convex surface is strictly convex and spacelike. These spacetimes are maximal domains of dependence essentially by construction. The rest of the statement of the proposition is simply descriptive. □

Labourie [52] has shown that the spacetimes in proposition 16 are foliated by spacelike convex surfaces of constant curvature; these surfaces are dual to surfaces of constant curvature in hyperbolic space. We conjecture that every de Sitter spacetime which is a small neighbourhood of a closed oriented spacelike hypersurface other than a sphere embeds in a unique one of the spacetimes in proposition 16. If the hypersurface is a torus then the projective structure is an affine structure; see [36] for a discussion of affine structure on tori. Projective structures on tori are still classified by the cotangent bundle of Teichmüller space, but the dimension is 2 rather than $6g-6$ and it is more natural to think of two affine structures as differing by a holomorphic differential than by a holomorphic quadratic differential. It is known that every homomorphism of the fundamental group of a closed surface of genus g to $\mathbf{SL}_2\mathbb{C}$ with image which is not conjugate into $\mathbf{SU}(2)$ nor into the stabilizer of a point or a pair of points in $\mathbb{CP}^1$ is the holonomy of a projective structure [42], and that given one projective structure there are infinitely many others with the same holonomy, but the map from the space of projective structures to the space of holonomy representations is not a covering map [47, 48].

However in the holomorphic description of projective structures, each fiber $\mathbb{C}^{3g-3}$ is mapped injectively by the holonomy to $\mathsf{Hom}(\pi_1 F, \mathbf{PSL}_2\mathbb{C})/\mathbf{PSL}_2\mathbb{C}$ [36]. See also [43], where projective structures whose holonomy is a Fuchsian group are classified, and [51] where projective structures with holonomy in $\mathbf{PGL}(2,\mathbb{R})$ are classified. Suppose we are given a closed spacelike surface with a neighbourhood modelled on de Sitter space. Gallo [42] actually shows that given a homomorphism from the fundamental group of a surface to $\mathbf{SL}_2\mathbb{C}$ with image not conjugate into $\mathbf{SU}(2)$ nor into the stabilizer of a point or pair of points in $\mathbb{CP}^1$, there is a decomposition of the surface into pairs of pants, such that the holonomy of any pair of pants is quasifuchsian. After an arbitrarily small change in the holonomy of each of the boundary circles, the eigenvalue $re^{i\theta}$ of the holonomy at a fixed point on $\mathbb{CP}^1$ will have θ a rational multiple of π. Then after passing to a finite cover of the surface using [11], we may assume that the surface has a decomposition along curves with purely hyperbolic holonomy into submanifolds S_i whose holonomy groups are quasifuchsian. The bending laminations of these submanifolds can be approximated by a union of long simple closed curves, so by another small change in holonomy we can assure that the submanifolds S_i can be decomposed into submanifolds with Fuchsian holonomy. So in trying to show that every closed spacelike surface with a neighbourhood modelled on de Sitter space lies in a domain of dependence of a projective structure, one may assume that the surface has a decomposition into pieces with Fuchsian holonomy. (There is however a difficulty with this approach which is pointed out in the proof of proposition 18.) Even under the assumption

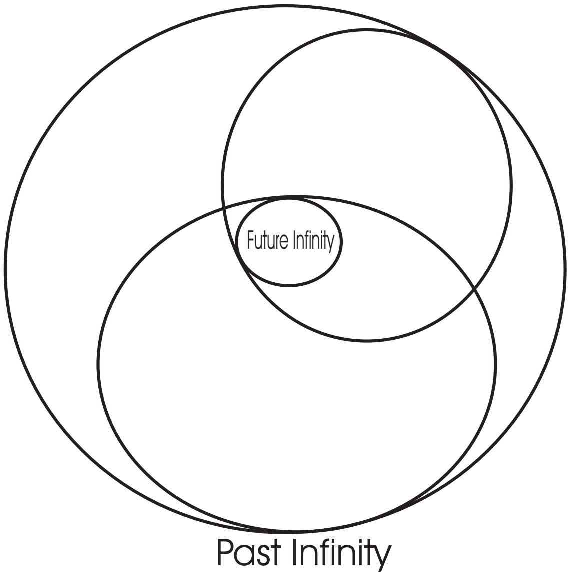

FIGURE 2. $1+1$-dimensional de Sitter space with some typical null geodesics

that the holonomy is conjugate into $\mathbf{SL}(2,\mathbb{R})$ it is not obvious that a closed spacelike surface with a neighbourhood modelled on de Sitter space is associated with a projective structure.

Let us consider the $1+1$ dimensional version of the problem. (Some information about the action of the automorphism group of de Sitter space is given in the proof of the next proposition without being explicitly stated in the proposition.)

Proposition 17. *Suppose M is a locally de Sitter $1+1$-dimensional spacetime which is a small oriented and orthochronous neighbourhood of a closed spacelike circle C. Either*

a) M embeds in a complete manifold M' such that the inclusion of C is a homotopy equivalence and M' has a compactification by a circle at future infinity and one at past infinity, or else

b) there is a hyperbolic element $\alpha \in \mathbf{SO}(2,1)_0$ such that α is the holonomy of C, and if P, Q are the attracting and repelling fixed points of α and R the third fixed point of α (R is the intersection of the tangents at P and Q to the invariant conic S^1 of $\mathbf{SO}(2,1)_0$) there is an arc PQ with endpoints P, Q on the invariant conic such that the line segments RP, RQ together with PQ bound an open disc PQR in de Sitter space, and M is an open subset of $PQR \cup PQ/\langle\alpha\rangle$: In short M embeds in a domain of dependence of a circle with $(\mathbf{PSL}_2\mathbb{R}, \mathbb{RP}^1)$-structure whose universal cover embeds in $\mathbb{RP}^1$ and whose holonomy is linear, or

c) M embeds in a domain of dependence $D(M)$ of a circle with $(\mathbf{PSL}_2\mathbb{R}, \mathbb{RP}^1)$-structure and unipotent holonomy such that the universal embeds in $\mathbb{RP}^1$. Then the universal cover $\widetilde{D(M)}$ of $D(M)$ is the complement in $\mathbb{RP}^2$ of the union of the closure $\mathsf{cl}\,\mathbb{H}^2$ and the tangent line to a point $P \in S^1 = \partial \mathsf{cl}\,\mathbb{H}^2$.

Proof. $1+1$-dimensional de Sitter spacetime X_{1+1} is the complement, in the double cover S^2 of $\mathbb{RP}^2$, of two antipodal round discs $D, -D$. Let ∂X_{1+1} be the boundary of

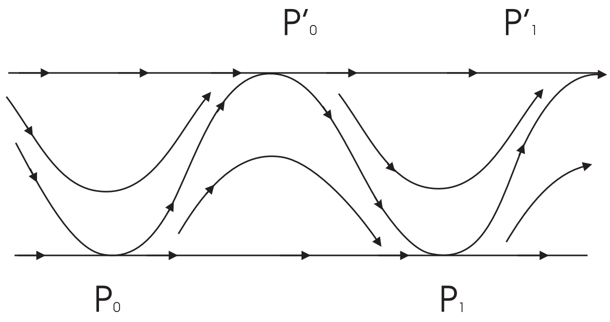

FIGURE 3. The universal cover of 1+1-dimensional de Sitter space and the flow lines of a unipotent subgroup

X_{1+1}: ∂X_{1+1} is the union of a circle at past infinity and one at future infinity. Note that X_{1+1} carries two rulings by null geodesics. A great circle on S^2 which is tangent to $D, -D$ at points $P, -P$ in S^2 is divided by $P, -P$ into two open semicircles. See Figure 2, in which stereographic projection has been used to draw X_{1+1} in the plane. Each semicircle is a null geodesic in de Sitter space. The null geodesics are divided into two disjoint families, the left and right rulings of X_{1+1}. Recall that if a manifold is locally modelled on a homogeneous space G/H with transition functions in G, the manifold is also locally modelled on the universal cover of G/H which is a homogeneous space with automorphism group in general a proper covering of G. The identity component of the isometry group of the universal cover $\widetilde{X_{1+1}}$ is the universal cover $\widetilde{\mathbf{SO}(2,1)_0}$ of $\mathbf{SO}(2,1)_0$. The holonomy of C is a well-defined element (up to conjugacy) $a \in \widetilde{\mathbf{SO}(2,1)_0}$ and acts on the boundary $\partial\widetilde{X_{1+1}}$. Suppose a acts on one (and therefore both) components of $\partial\widetilde{X_{1+1}}$ without fixed points. Then any orbit is unbounded above and below in $\mathbb{R} = \partial\widetilde{X_{1+1}}$. Then the projection of $\mathsf{dev}\,\widetilde{C}$ down say the left null ruling to either of the boundary components must be a submersion onto. Then given any point in $\widetilde{X_{1+1}} \cup \partial\widetilde{X_{1+1}}$, either the future pointing or the past pointing timelike and null rays meet $\mathsf{dev}\,\widetilde{C}$ in a compact interval, and we deduce that a acts properly discontinuously on $\widetilde{X_{1+1}} \cup \partial\widetilde{X_{1+1}}$. Thus M embeds in $M' = \widetilde{X_{1+1}}/\langle a\rangle$. In particular if the image of the holonomy is trivial in $\mathbf{SO}(2,1)_0$, then C is embedded in a finite cover of X_{1+1} and the complete manifold M' is that finite cover.

On the other hand, if a has a fixed point $P = P_0$ on say the past component of $\partial\widetilde{X_{1+1}}$, then all the translates P_n, $n \in \mathbb{Z}$ of P by the deck group of the covering $\partial\widetilde{X_{1+1}} \to \partial X_{1+1}$ are also fixed. The lifts of the antipodal points are also fixed: These are points P'_n on the future boundary such that there is a null geodesic in the left ruling joining P_n to P'_n and a null geodesic in the right ruling joining P'_n to P_{n+1}. Now we distinguish two cases. Either a is the lift of a hyperbolic element of $\mathbf{SO}(2,1)_0$ or else a is the lift of a unipotent element of $\mathbf{SO}(2,1)_0$. See Figure 3 and Figure 4. In the hyperbolic case, there are fixed points Q_n, Q'_n with Q_n between P_n and P_{n+1}, Q'_n between P'_n and P'_{n+1}. There are also fixed points R_n on the intersections of the null geodesics from Q_n to Q'_n and the null geodesic from P'_n to P_{n+1}. Let $p_L : \widetilde{X_{1+1}} \cup \partial\widetilde{X_{1+1}} \to \partial\widetilde{X_{1+1}}$ be the projection down the left ruling

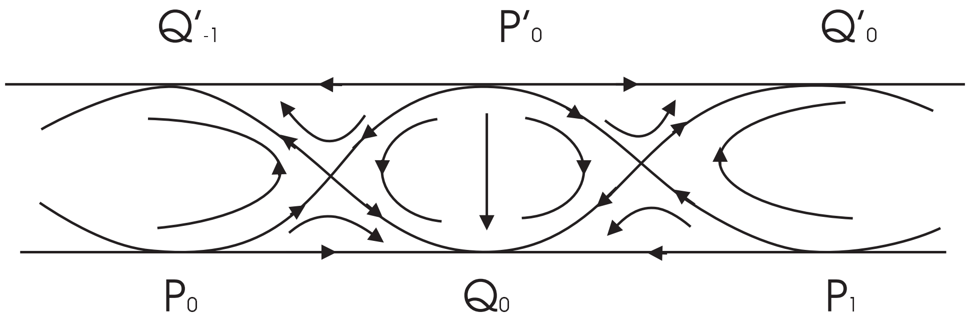

FIGURE 4. The universal cover of 1+1-dimensional de Sitter space and the flow lines of a hyperbolic subgroup

to the past boundary. Because $\widetilde{C}$ is spacelike $p_L|_{\widetilde{C}}$ is a homeomorphism onto its image. Since p_L intertwines the action of $\pi_1 C$ on $\widetilde{C}$ with the action of a on the past component of $\partial\widetilde{X_{1+1}}$, $p_L(\widetilde{C})$ is an open interval containing no fixed point of a. Therefore $p_L(\widetilde{C})$ is a bounded interval. The endpoints of the interval $p_L(\widetilde{C})$ are fixed by a. So the interval must be (up to a deck transformation and possibly interchanging the labels P, Q) the interval P_0, Q_0. Then the projection $p_R(\widetilde{C})$ must be one of the intervals P_0Q_0, Q_0P_1, P_1Q_1. In the first of these three cases C lies in the domain of dependence of the quotient of the interval P_0Q_0 by $\langle a\rangle$; the universal cover of this domain of dependence is the open triangle $P_0R_0Q_0$, and there is a partial compactification by adding the quotient of the interval P_0Q_0. Similarly in the third case C lies in a future complete domain of dependence. The second case cannot arise because any $\langle a\rangle$ orbit in the open quadrangle $Q_0R_0P_0'R_1$ (whose boundary consists of null geodesic segments) accumulates at P_0' and Q_0 which is impossible since $\widetilde{C}$ is spacelike.

The unipotent case is similar but a little simpler. □

Proposition 18. *Let M be a de Sitter manifold which is a small neighbourhood of a closed spacelike torus. Then (perhaps after a change of time orientation) there is a torus with affine structure such that M embeds in the spacetime associated to the torus by proposition 16.*

Proof. First of all, the holonomy of M cannot be conjugate into the subgroup $\mathbf{SU}(2)$ of $\mathbf{SL}_2\mathbb{C}$. If it was, then projection onto $\mathbb{CP}^1$ along the family of lines which meet at the fixed point inside $\mathbb{H}^3$ of the subgroup conjugate to $\mathbf{SU}(2)$ would define a Riemannian structure on T of constant positive curvature, which is impossible for a closed surface of positive genus. Suppose we show that after an arbitrarily small change of the geometric structure on a neighbourhood of T, the resulting manifold embeds in a domain of dependence of some torus at future infinity having an affine structure. Then the manifold M is associated to an affine structure by proposition 16 because the space of all de Sitter structures on a germ of a neighbourhood of T is a union of components each of which is mapped by the holonomy submersively to the space of conjugacy classes of representations of the fundamental group, and the map from the space of affine structures to the universal cover of the space of conjugacy classes of representations of $\mathbb{Z}\oplus\mathbb{Z}$ which are not conjugate into $\mathbf{SU}(2)$

is proper, so if arbitrarily near geometric structures embed in a domain of dependence then so does the given geometric structure on M. However, the properness may fail in the case of the nonelementary representations of the fundamental group of a surface of genus greater than one. Also, if a finite cover of M embeds in a domain of dependence of an affine structure, M itself does. This applies equally to the question of surfaces of genus greater than one. We can then suppose that the holonomy is hyperbolic or loxodromic with rotational angle a rational multiple of π, rather than unipotent. So after passing to a finite cover of the spacetime we may assume that the holonomy is conjugate into the identity component of the diagonal subgroup of $\mathbf{SL}_2\mathbb{R}$. Then by another small change in the geometric structure we have (in an appropriate basis) one of the summands of $\mathbb{Z} \oplus \mathbb{Z}$ is in the kernel of the holonomy. Now an infinite cyclic cover T' of T lies in the simply connected de Sitter space, which is the complement in the three-sphere of two antipodal round discs. There is an isometric S^1 action on de Sitter space which commutes with the holonomy of T and fixes an axis, that is, the great circle which passes through the two fixed points of the action of the holonomy of T on either of the spheres at infinity of de Sitter space. There is a global section to this action so on T there is an S^1-valued function say θ defined everywhere except at finitely many points which are the images in T of the intersection of T' with the axis. Euler characteristic considerations imply that this finite set is empty. Now consider the submanifold T_0 of T which has a fixed value of θ. Its preimage in T' is the preimage by the developing map of a plane (i.e., an equatorial two-sphere) in de Sitter space through the axis. Each component of T_0 develops into a spacelike curve. Proposition 17 applies and we conclude that (in the $\mathbb{RP}^3$ model) T' lies between the tangent planes to the sphere at infinity at the two points fixed by the holonomy of T, from which it follows readily that M embeds in a domain of dependence. $\square$

7. Anti de Sitter manifolds

As with de Sitter space, we will work with a synthetic description of anti de Sitter space. Consider the quadric $Q : AD - BC = 0$ in $\mathbb{RP}^3$. It divides $\mathbb{RP}^3$ into two regions, $AD - BC > 0$ and $AD - BC < 0$. The quadric is the Segre embedding of $\mathbb{RP}^1 \times \mathbb{RP}^1$ in $\mathbb{RP}^3$: $((X_1 : Y_1), (X_2 : Y_2)) \mapsto (X_1X_2 : X_1Y_2 : Y_1X_2 : Y_1Y_2)$. (The Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$ comes from projectivizing the canonical map $V \times W \to V \otimes W, (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \otimes \mathbf{w}$, where $\mathbb{P}^n, \mathbb{P}^m$ are the projectivizations of vector spaces V, W.) Thus there are two families, the left and right rulings, of straight lines on Q. Through each point passes one line of each ruling. Any line of the left ruling intersects each line of the right ruling in exactly one point, and vice versa. So the lines of the right ruling are parametrized by the points of any one line of the left ruling, and vice versa. Now recall that in $\mathbb{R}^4 = \mathbb{R}^2 \otimes \mathbb{R}^2$, the decomposable vectors are those that lie on the quadric $AD = BC$. The subgroup G_1 of $\mathbf{GL}(\mathbb{R}^4)$ which preserves the decomposable vectors is the group which preserves the quadratic form $AD - BC$ up to scale. G_1 has an index two subgroup $G_2 = \mathbf{GL}(\mathbb{R}^2) \times \mathbf{GL}(\mathbb{R}^2)/\langle (aI, I) = (I, aI) \rangle$; the non-identity coset of G_2 in G_1 is generated by an involution which exchanges the left and right factors of the tensor product $\mathbb{R}^4 = \mathbb{R}^2 \otimes \mathbb{R}^2$. G_2 contains as a normal subgroup, with quotient group $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{R}^*_{>0}$ (where $\mathbb{R}^*_{>0}$ is the multiplicative group of the positive reals), the group $\mathbf{SO}(2,2)_0 = \mathbf{SL}_2\mathbb{R} \times \mathbf{SL}_2\mathbb{R}/(-I, I) = (I, -I)$. So the identity component of the automorphism group of the quadric Q is $\mathbf{SO}(2,2)_0/\langle -I \rangle = \mathbf{PSL}_2\mathbb{R} \times \mathbf{PSL}_2\mathbb{R}$.

The action of the left factor preserves each line $L_{(\lambda:\mu)} = \{(A : B : C : D) \mid (A : B) = (C : D) = (\lambda : \mu)\}$ while the right action preserves each line $R_{(\lambda:\mu)} = \{(A : B : C : D) \mid (A : C) = (B : D) = (\lambda : \mu)\}$. We call the region $AD - BC > 0$ *anti de Sitter space*. It carries a canonical Lorentzian structure, and we let X denote anti de Sitter space with its Lorentzian structure. Given a point x of anti de Sitter space, draw the tangent cone to the quadric Q. This determines a Lorentzian quadratic form on the tangent space at x, up to scale: The tangents to Q are in the null directions. In fact the scale of the metric is also determined. Given a spacelike tangent vector at x, the straight line through x meets Q in two points E, H. Given two points F, G in anti de Sitter space on the line l, the distance $d(F, G)$ can be defined in terms of the cross-ratio of E, F, G, H but we will not need the explicit formula.

Letting F, G approach x one defines the length of a tangent vector at x. (The lengths of timelike vectors can be defined by complexifying so that every line is tangent to Q or else meets Q in two distinct points, possibly imaginary.)

This description of a geometry in terms of projective space with a distinguished quadric is due to Cayley, in the case of elliptic and hyperbolic geometry. See [37]. See [38] for the Lorentzian case.

We may identify anti de Sitter space with $\mathbf{PSL}_2\mathbb{R}$ by identifying $(a : b : c : d)$ with its representative, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ unique up to multiplication by -1, with $ad - bc = 1$. Then the left and right actions of $\mathbf{PSL}_2\mathbb{R}$ are by left and right matrix multiplication. The Killing form is bi-invariant, which gives an alternative construction of the Lorentzian pseudometric. So we have a model of the Lorentzian symmetric space $\mathbf{PSL}_2\mathbb{R}$ in which geodesics are represented by straight lines in $\mathbb{RP}^3$. Totally geodesic hyperplanes are the intersections of planes with de Sitter space. So they are given by linear equations $P_{(e:f:g:h)} = \{(a : b : c : d) : ae + bf + cg + dh = 0\}$. A plane is spacelike, null or Lorentzian according as $eh - gf > 0$, $= 0$, or < 0. All the geodesics which are normal to $P_{(e:f:g:h)}$ meet in the point $(e : f : g : h)$. In particular if $P_{(e:f:g:h)}$ is a null plane, $P_{(e:f:g:h)}$ is tangent to Q at $(e : f : g : h)$. Each spacelike plane is isometric to the hyperbolic plane, each null plane has a unique ruling by null lines and the metric is a transverse metric to this foliation, and each Lorentzian plane is topologically a Möbius strip and is isometric to $1+1$-dimensional de Sitter space. Now let M be a Lorentzian manifold locally isometric to anti de Sitter space and containing a closed spacelike hypersurface S with an orthochronous neighbourhood. There is a development map $d : \widetilde{S} \to \mathbf{PSL}_2\mathbb{R}$. Given a future pointing null direction $\mathbf{u}$ at $x \in \widetilde{S}$, there is a null geodesic at $d(x)$ with tangent vector $d_*\mathbf{u}$ where d_* is the derivative of d. This null geodesic is tangent to Q at a point lying on a line $L_{a(\mathbf{u})}$ of the left ruling and a line $R_{b(\mathbf{u})}$ of the right ruling. This defines two trivializations of the bundle of future pointing null directions. Now the lines of the left, respectively the right rulings are permuted by the right, respectively the left action of $\mathbf{PSL}_2\mathbb{R}$.

Proposition 19. *The holonomy $\rho = (\rho_L, \rho_R) : \pi_1 S \to \mathbf{PSL}_2\mathbb{R} \times \mathbf{PSL}_2\mathbb{R}$ satisfies the condition that ρ_L, ρ_R have Euler class $2 - 2g$ and therefore are isomorphisms to Fuchsian groups.*

Proof. As in proposition 1, the unit tangent bundle may be identified with the bundle of future pointing null directions. So the maps a, b introduced above identify $UT(S)$ with the flat $\mathbb{RP}^1$-bundles associated with $\rho_R : \pi_1 S \to \mathbf{PSL}_2\mathbb{R}$, respectively

$\rho_L : \pi_1 S \to \mathbf{PSL}_2\mathbb{R}$. So each of ρ_L, ρ_R has Euler class $2-2g$ and by Goldman's theorem ρ_L, ρ_R are isomorphisms to cocompact Fuchsian groups. $\square$

So the holonomy of a spacelike surface S in an anti de Sitter spacetime can be considered as a point of $\mathsf{Teich}(S) \times \mathsf{Teich}(S)$. (Note the analogy with quasifuchsian groups). Now we will show that any point in $\mathsf{Teich}(S) \times \mathsf{Teich}(S)$ may be attained.

For the moment let us distinguish G_L, the subgroup of the identity component $\mathsf{Aut}_0 X$ which acts by matrix multiplication on the left, from G_R, the subgroup which acts by matrix multiplication on the right. We will want to think of $\mathsf{Aut}_0 X$ acting on the right of X. Thus the action $G_L \times G_R \times X \to X$ is given by $(g, h, x) \mapsto g^{-1} \cdot x \cdot h$. Recall that G_L permutes the lines of the right ruling, which are parametrized by a projective line $\mathbb{RP}^1_L$ so G_L acts on $\mathbb{RP}^1_L$ and similarly G_R acts on $\mathbb{RP}^1_R$. Now consider the intersection $P \cap Q$ of the quadric Q with a spacelike plane P. $P \cap Q$ is a conic and determines a one-to-one correspondence between the set of lines of the left ruling and the set of lines of the right ruling, since $P \cap Q$ meets each line in one point. So $\mathbb{RP}^1_L$ and $\mathbb{RP}^1_R$ can be identified with $P \cap Q$. Under this identification G_L and G_R are each identified with the identity component of the group of projectivities of the plane P which preserve $P \cap Q$. So P determines an isomorphism $\phi : G_R \to G_L$. The graph of this automorphism is a subgroup $\mathsf{Aut}_P X = \{\alpha \in \mathsf{Aut}_0 X : \alpha(x) = \phi(g)^{-1} \cdot x \cdot g\}$ of $\mathsf{Aut}_0 X$. If we identify G_L, G_R with $\mathbf{PSL}_2\mathbb{R}$ using matrix multiplication, $\phi(g) = aga^{-1}$ for some $a \in \mathbf{PSL}_2\mathbb{R}$. If $\alpha \in \mathsf{Aut}_P X$, $\alpha(x) = ag^{-1}a^{-1}xg$; that is, $a^{-1}x$ is conjugated, so a is fixed by α. Thus each $\mathsf{Aut}_P X$ is the stabilizer of a certain point $a(p) \in X$; $a(P)$ is the dual point of P, i.e., the intersection of all the normal geodesics to $P \cap X$.

Now given two homomorphisms $\rho_L : \pi_1 S \to G_L = \mathsf{Aut}_0 \mathbb{RP}^1_L$, $\rho_R : \pi_1 S \to G_R = \mathsf{Aut}_0 \mathbb{RP}^1_R$, each with Euler class $2-2g$, there is a homeomorphism $h : \mathbb{RP}^1_L \to \mathbb{RP}^1_R$ conjugating ρ_L to ρ_R, because $\rho_L(\pi_1 S), \rho_R(\pi_1 S)$ are isomorphic cocompact Fuchsian groups and so their actions on their limit circles are topologically conjugate. Moreover, if $\mathbb{RP}^1_L$ and $\mathbb{RP}^1_R$ are oriented so that the map $i_p : \mathbb{RP}^1_L \to \mathbb{RP}^1_R$ determined by a spacelike plane P is orientation preserving, the homeomorphism h is orientation preserving. We may consider the graph of h as embedded in Q, as $Q \cong \mathbb{RP}^1_L \times \mathbb{RP}^1_R$. Q has a conformally Lorentzian structure. Because h is orientation preserving, nearby points on the graph of h are spacelike with respect to each other. (For an orientation reversing homeomorphism, nearby points would be timelike with respect to each other. The intersection of Q with a timelike plane is the graph of an element of the non-identity component of $\mathbf{PGL}_2\mathbb{R}$, if a spacelike plane has been used to identify $\mathbf{PSL}_2\mathbb{R}_L$ and $\mathbf{PSL}_2\mathbb{R}_R$.) So given 3 points A, B, C on $\mathsf{graph}\,(h)$, the plane through A, B, C is spacelike.

Lemma 5. *There is a plane (necessarily spacelike) disjoint from* $\mathsf{graph}\,(h) \subset Q$.

Proof. Given 3 points A, B, C on the graph of h let A', B', C' be the intersections of lines l_A, l_B, l_C of the left ruling with lines r_B, r_C, r_A respectively of the right ruling, and let A'', B'', C'' be the intersections of l_A, l_B, l_C with r_C, r_A, r_B. Then the graph of h lies in the three twisted quadrilaterals $AA'BB'', BB'CC'', CC'AA''$ on Q. The configuration is unique up to a projective transformation and can be realized in $\mathbb{R}^3$ with Q given by $x^2 + y^2 = z^2 + 1$ and

$$A = (1, 0, 0), B = (\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}, 0), C = (\cos\frac{2\pi}{3}, -\sin\frac{2\pi}{3}, 0),$$
$$A' = (2\sin\frac{\pi}{3}, 2\cos\frac{\pi}{3}, \sqrt{3}), B' = (-2, 0, \sqrt{3}), C' = (-2\sin\frac{\pi}{3}, 2\cos\frac{\pi}{3}, \sqrt{3})$$

and A'', B'', C'' are the reflections of C', A', B' in the plane $z = 0$. Then there are many spacelike planes, including the plane at infinity and the planes $z = k$ where $|k| > \sqrt{3}$, disjoint from $\mathsf{graph}\,(\phi)$. □

Having chosen a spacelike plane P_∞, the convex hull $X(\phi)$ of $\mathsf{graph}\,(\phi)$ in $\mathbb{RP}^3 - P_\infty$ is defined, and $\pi_1 S$ acts on $X(\phi)$ by $\rho = (\rho_L, \rho_R) : \pi_1 S \to G_L \times G_R$. From the fact that $\mathsf{graph}\,(h)$ is a spacelike curve in the conformally Lorentzian structure on Q it follows that lines and planes in the boundary of the convex hull are all spacelike.

Proposition 20. *Given a point (x, y) in $\mathsf{Teich}(S) \times \mathsf{Teich}(S)$ there exists a compact spacetime $X(x, y)$ with boundary, homeomorphic to $S \times [0, 1]$ by a homeomorphism taking any $S \times \{x\}$ to a closed spacelike hypersurface, which is locally modelled on anti de Sitter space, and which has left and right holonomies ρ_L, ρ_R determined by the point (x, y). Moreover the boundary is spacelike, locally convex and has no extreme points and these conditions uniquely determine $X(x, y)$ up to Lorentz isometry.*

Proof. Choose homomorphisms $\rho_L, \rho_R : \pi_1 S \to \mathbf{PSL}_2\mathbb{R}$ representing $(x, y) \in \mathsf{Teich}(S) \times \mathsf{Teich}(S)$. We have seen that there exists a compact convex set $X(\phi)$, equivariant with respect to $(\rho_L, \rho_R) : \pi_1 S \to G_L \times G_R = \mathsf{Aut}_0 X$. The boundary of $X(\phi)$ in Q is the graph of a homeomorphism conjugating ρ_L to ρ_R, and $F(\phi), P(\phi)$ are the future and past boundary components of $X(\phi)$. $F(\phi), P(\phi)$ are locally convex and spacelike.

We need to see that $\rho(\pi_1 S)$ acts properly discontinuously on $F(\phi)$. Here is one way to see this; proposition 21 will give another. Given $p \in F(\phi)$ and a sequence of distinct group elements $\gamma_i \in \pi_1 S$, after passing to a subsequence, there exists $x \in \mathbb{RP}^1_R$ such that $\rho_R(\gamma_i y) \to x \in \mathbb{H}^2 \cup \mathbb{RP}^1_L$, uniformly for y in a compact subset of $\mathbb{H}^2$. Fix a spacelike supporting plane T through p. (If the supporting plane is unique, it is the tangent plane.) There is a rotationally invariant measure on the rays from T, and this determines a measure $\mu^L_{p,T}$ on $\mathbb{RP}^1_L$ and a measure $\mu^R_{p,T}$ on $\mathbb{RP}^1_R$. Now $\rho_L(\gamma_i)_*\mu^L_{p,T} \to \delta_{\phi(x)} \in M(\mathbb{RP}^1_L)$ and $\rho_R(\gamma_i)_*\mu^R_{p,T} \to \delta_x \in M(\mathbb{RP}^1_R)$ where $M(\mathbb{RP}^1_L), M(\mathbb{RP}^1_R)$ are the spaces of probability measures on $\mathbb{RP}^1_L, \mathbb{RP}^1_R$ respectively and where $\delta_{\phi(x)}, \delta_x$ are the point masses at $\phi(x)$ and x respectively. Now suppose that $\rho(\pi_1 S)$ did not act properly discontinuously on $F(\phi)$. Then for some p, there is a sequence γ_i such that $\rho(\gamma_i) \cdot p$ remains within a compact set $K \subset F(\phi)$. But then the measures $\rho_L(\gamma_i)_*\mu^L_{p,T}$ lie in a bounded set in the space $C^0(\mathbb{RP}^1_L)$, the space of measures with a continuous density with respect to the $\mathbf{SO}(2)$-invariant measure, and similarly for $\rho_R(\gamma_i)_*\mu^R_{p,T}$. So $\rho(\pi_1 S)$ acts properly discontinuously on $F(\phi)$.

To deduce that the action on $X(\phi)$ is properly discontinuous, observe that if $p \in X(\phi)$, the future pointing null and timelike rays from p meet $F(\phi)$ in a compact disc $D(p)$. $D(p)$ depends continuously on p.

Now for the uniqueness of $X(\phi)$.

Lemma 6. *Suppose $f : A \to X$ is a locally isometric map of a complete connected Riemannian manifold A to anti de Sitter space. Then a) f is proper; b) f is an embedding, and $f(A)$ intersects every timelike geodesic in only one point.*

Proof. Fix an action of $\mathbf{SO}(2)$ on X by right multiplication, and consider the Riemannian submersion $p : X \to X/\mathbf{SO}(2) = \mathbb{H}^2$. (The preimage by p of a round disc in the hyperbolic plane is a solid torus with boundary a doubly ruled hyperboloid; one of the rulings is by orbits of the $\mathbf{SO}(2)$-action.) The derivative of p can only lengthen a spacelike vector. (Note however that X does not contain a subset which is mapped isometrically to $\mathbb{H}^2$ by p, because the field of planes orthogonal to the $\mathbf{SO}(2)$-action is totally non-integrable. In particular the restriction of p to a totally geodesic spacelike hypersurface, which is isometric to $\mathbb{H}^2$ is not an isometry.) So if A is complete, the projection must be a covering map. Since $\mathbb{H}^2$ is simply connected, A is diffeomorphic to $\mathbb{H}^2$, f is proper. Given any timelike geodesic, there is a unique $\mathbf{SO}(2)$-action by right multiplication of which it is an orbit, so *b*) follows. □

Now we identify A with its image, and consider the closure $\mathsf{cl}\, A$ in $\mathbb{RP}^3$ of $f(A)$. We want to show $\mathsf{cl}\, A$ is a disc.

In lemma 7, we consider that a null geodesic $l \cap X$ does not contain its point at infinity, the point of tangency $l \cap Q$ of tangency of the line l with Q. Thus a point x divides a null geodesic into two half lines, the past and future pointing null rays through x, each of which is homeomorphic to $[0, \infty]$.

Lemma 7. *a) Given a spacelike disc D which meets no timelike or null geodesic in more than one point, let $F(D)$, respectively $P(D)$, be the set of $x \in X$ such that every past pointing, respectively future pointing null ray or timelike geodesic through x meets D. Then $P(D) \cap F(D) = D$. b) Given a complete spacelike submanifold $A \subset X$, the closure of A in $X \cup Q \subset \mathbb{RP}^3$ is a disc $A \cup \partial A$ and the boundary $\partial A = \mathsf{cl}\, A \cap Q$ is a circle which is nowhere timelike in the conformally Lorentzian structure on Q.*

Before proving the lemma, let us amplify, perhaps pedantically, the statement of part *b*). A differentiable curve on Q is nowhere timelike if its tangent vector is nowhere timelike in the conformally Lorentzian structure on Q. In general, a nowhere timelike curve is a curve that can be approximated by differentiable nowhere timelike curves. An alternative definition may be given without recourse to differentiability: A curve $C : t \to C(t)$ on Q is nowhere timelike if given any point $E = C(s)$ on C and any sufficiently small pair $EFGH, EJKL$ if quadrangles on Q such that JEF, HG, KL are short segments of lines of the left ruling and GF, HEJ, JK are short segments of lines of the right ruling, $C(t)$ lies in the union of the two small quadrangles $EFGH, EJKL$ for any t sufficiently close to s. See Figure 5. Note that given any two points on Q which do not lie on a line in one of the two rulings, there is a spacelike geodesic in X with the two given points as endpoints.

Proof. *a*) If there was a point $y \in P(D) \cap F(D)$ such that y is not in D then any null geodesic through y meets D in two points. This proves *a*). Now let $H(D) = P(D) \cup F(D)$.

Sublemma 1. *Given any complete spacelike surface $A \subset X$, there is a spacelike plane in X whose closure in $X \cup Q \subset \mathbb{RP}^3$ is disjoint from the closure of A.*

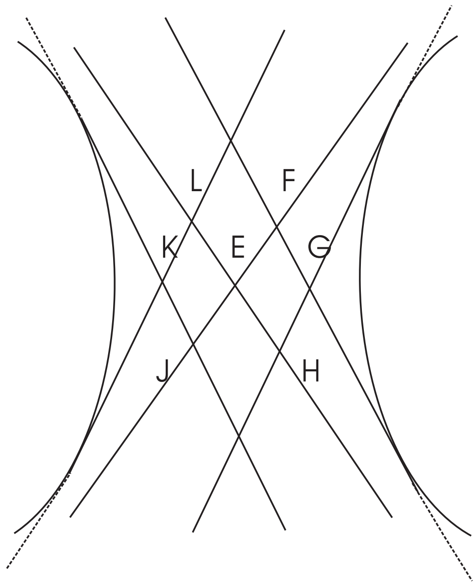

FIGURE 5. A nowhere timelike curve on the quadric at infinity

Proof. Given $p \in A$ consider the dual plane to p, which meets Q in the points of tangency of the null lines through p. Every normal to the dual plane passes through p, so if A met the dual plane there would be a timelike geodesic which met A in two distinct points. To see that the dual plane is disjoint from the closure of A in $\mathbb{RP}^3$, assume without loss of generality that A contains a neighbourhood of p which lies in a plane. The union of the dual planes is a neighbourhood of the dual plane of p. □

Thus we can choose a plane at infinity, and then the convex hull of the closure of A in $\mathbb{RP}^3$ is compact. Furthermore, there is a compact convex set which is an intersection of half-spaces with spacelike boundaries, and contains A; let $B = B(A)$ be the minimal such set. A half-space with spacelike boundary can either contain the immediate past of its boundary in which case we will call it "past-complete" or not; the boundary of B divides into the future boundary where every point has at least one spacelike supporting plane which B lies in the past of, the past boundary, and the rest.

Fix a point $x \in X$. Let D_r be the disc in $\mathbb{H}^2 = X/\mathbf{SO}(2)$ with radius r and center $p(x)$, and let $T_r = p^{-1}D_r$ be its preimage in X and $A_r = A \cap T_r$. There is an $\mathbf{SO}(2) \times \mathbf{SO}(2)$ action on $X \cup Q$ whose orbits are the tori T_r. A spacelike line through x together with this $\mathbf{SO}(2) \times \mathbf{SO}(2)$ action determines identifications of the tori T_r with Q. Choose a sequence $r_n \to \infty$ such that ∂A_{r_n} converges in the space of compact subsets of $\mathbb{RP}^3$ with the Hausdorff topology, or equivalently, in the Hausdorff topology on the space of compact subsets of Q using the identifications of the tori T_r with Q. Each ∂A_{r_n} can be considered as the graph of a homeomorphism between the left and right copies of $\mathbf{SO}(2)$. So the relation ∂A_∞ can be approximated by homeomorphisms. It follows that ∂A_∞ intersects each line on Q

in a nonempty connected set. It follows that ∂A_∞ is a nowhere timelike topological circle. There is a plane disjoint from ∂A_∞ by (the argument of) lemma 5. Consider the sets $H(A_r)$. Since this is an increasing family, parametrized by $[0, \infty]$, of compact sets any subsequence has the same Hausdorff limit. Let us define $H(A_\infty)$ to be the set of points which have dual planes which do not cross the circle A_∞. (The circle A_∞ separates a small neighbourhood of itself in Q. Let $N(y)$ be the intersection of the dual plane of y with Q. $N(y)$ crosses A_∞ at $w \in A_\infty$ if $N(y)$ contains arbitrarily small intervals containing w which meet both components of the complement of A_∞ in the small neighbourhood of A_∞.) Then the closed set $H(A_\infty)$ contains all the sets $H(A_r)$. For if $y \in H(A_r)$ the null cone of y intersects A in a compact set, and so the conic say $N(y)$ of tangency of the null cone with Q can't cross A_∞ because if $N(y)$ did then the null cone would intersect A_{r_n} for all sufficiently large n. Since $H(A_\infty \cap Q) = A_\infty$, the Hausdorff limit of the sets $H(A_r)$ is $H(A_\infty) = \partial A$ and also the closure of A in $X \cap Q$ is $A \cap \partial A$. Now to see that $\mathsf{cl}\, A = A \cap \partial A$ is a disc, consider the action of the diagonal subgroup of $\mathbf{SO}(2) \times \mathbf{SO}(2)$ on $X \cap Q$. It gives a continuous bijection of $\mathsf{cl}\, A$ onto the disc. Since $\mathsf{cl}\, A$ is compact this is a homeomorphism. $\square$

The following sublemma, the proof of which follows a suggestion of B. Bowditch and N.H. Kuiper, can be used to simplify the proof of lemma 7. It is of some independent interest; perhaps it is well-known.

Sublemma 2. *Given an immersion $i : D^n \to \mathbb{RP}^n$ of a disc such that the boundary is strictly locally convex, i is an embedding and the image is a convex set in the complement of some hyperplane.*

Proof. It follows from the local convexity that any two points can be joined by a distance minimizing geodesic. i is an embedding because if $p, q \in D^n$ and $i(p) = i(q)$, then for any r in ∂D^n such that $i(r) \neq i(p)$ the geodesics from r to p, q must have a common initial segment, which is in the interior of D^n by local convexity. So one (say p) of p, q is in the interior. As every geodesic ray through p is the preimage of a closed geodesic, every geodesic ray through p eventually reaches the boundary or else covers a closed geodesic. Since every ray through p which reaches the boundary passes through q, possibly after extension through p, either there is a closed geodesic or else an open set of geodesic rays through p which reach q. The set of geodesics from p which reach q must be open as well as closed because of the convexity of the boundary. It follows that all geodesics from p reach q and then return to p, so D^n has no boundary, which is absurd. Now consider D^n as embedded in the double cover of $\mathbb{RP}^n$ and consider the cone over D^n in $\mathbb{R}^{n+1}$; it is strictly convex and it follows that D^n lies on one side of a hyperplane and is a convex set in the affine space defined as the complement of that hyperplane. $\square$

Here is an alternative proof of lemma 7, worked out with Bowditch. ∂T_r is an orbit of $\mathbf{SO}(2) \times \mathbf{SO}(2)$ and therefore is a doubly ruled quadric. So the interior of T_r has a Lorentzian structure defined by ∂T_r. In this Lorentzian structure, light cones are narrower than in the Lorentzian structure of X, so more vectors are spacelike. So the intersection $A_r = A \cap T_r$ is spacelike in the new Lorentzian structure. After choosing a different product structure on $\mathbf{SO}(2) \times \mathbf{SO}(2)$ the intersections A_r become graphs of functions all of which have Lipschitz constant 1. So it suffices to show pointwise convergence at one point in order to show that the sets A_r converge to a nowhere timelike circle. It is easy to see that the intersection of A with any indefinite plane has closure an arc, and the pointwise convergence follows. This proof is shorter, but it seems useful to understand the sets $H(A_r)$ occurring in lemma 7.

Extending the definitions of $F(D), P(D), H(D)$ given in lemma 7 and its proof to the case of a noncompact spacelike surface, we see that $H(A) = H(A_\infty)$ where $H(A_\infty)$ has just been defined. Suppose y lies in the interior of the convex hull of ∂A. Then no point in the dual plane of y can be in $H(A)$. For if z is in the dual plane of y, then the dual plane of z passes through y, and so meets Q in a conic which crosses ∂A. Thus there is a plane disjoint from $H(A)$, except in the case that A is a spacelike plane. In any case there is a plane disjoint from the interior of $H(A)$. Supposing there is a plane disjoint from $H(A)$, $H(A)$ is the intersection of all the closed half-spaces with boundary a totally geodesic null plane which contain A. So $H(A)$ is a compact convex set, except when A is a plane in which case $H(A)$ is the union of a closed convex cylinder and one point at infinity. We can also think of $H(A)$ as the closure of the region in which any spacelike surface with boundary ∂A must lie. Now given a compact Lorentzian manifold Z, locally isometric to anti de Sitter space, with spacelike locally convex boundary with no extreme points, with the holonomy of $X(\phi)$, the manifold can be slightly thickened to Z' to make the boundary smooth. Then lemma 7 applies, and we deduce that the closure of the universal cover of $\partial Z'$ is the union of $\mathsf{graph}\,(\phi)$ and two locally convex spacelike surfaces. Alternatively lemma 7 could have been proven for spacelike surfaces which were not smoothly immersed. It follows that the universal cover of either boundary component of Z is one of the two boundary components of $X(\phi)$. So the boundary is $F(\phi) \cup P(\phi)$. It follows that Z is the quotient of $X(\phi)$. $\square$

Proposition 21. *Given an orthochronous anti de Sitter spacetime which contains a product neighbourhood $N(F)$ of a closed spacelike surface F, then there is a smaller neighbourhood $N'(F)$ which embeds in $\mathsf{int}\big(H(F(\phi))\big)/\pi_1 F$ where ϕ conjugates the left and right holonomies of $\pi_1 F$. If $N(F)$ is a domain of dependence then $N(F)$ embeds in $\mathsf{int}\big(H(F(\phi))\big)/\pi_1 F$. So $\mathsf{int}\big(H(F(\phi))\big)/\pi_1 F$ is a domain of dependence which contains all other domains of dependence with the same holonomy.*

Proof. First we show that F has negative Euler characteristic. F cannot be a sphere or projective plane, because anti de Sitter space contains no spacelike sphere. If F had Euler characteristic 0, then F or a double cover is a torus; suppose F is a torus. By lemma 6 the holonomy cover $\widehat{F}$ of F is embedded and simply connected. We could argue that $\widehat{F}$ is quasi-isometric to a hyperbolic plane and so the ball of radius r in $\widehat{F}$ grows exponentially with r, while the cover of a torus has only polynomial volume growth. Alternatively, the left holonomy defines a homomorphism from $\mathbb{Z}\oplus\mathbb{Z}$ into an abelian subgroup, necessarily a 1-parameter subgroup, of G_L. So after an arbitrarily small change in holonomy the image is cyclic. The right holonomy is topologically conjugate to the left holonomy so the holonomy has kernel which contradicts lemma 6. So F has negative Euler characteristic.

We know that there is a manifold $X(\phi)/\pi_1 F$ with the same holonomy as $N(F)$, and it is easy to see that $\pi_1 F$ acts properly discontinuously on the enlargement $\mathsf{int}\big(H(F(\phi))\big) = \mathsf{int}\big(H(P(\phi))\big) = \mathsf{int}(H(\mathsf{graph}\,\phi))$. The universal cover of F isometrically immerses in anti de Sitter space, equivariantly with respect to the action of the holonomy of $X(\phi)/\pi_1 F$, and we know that this immersion is an embedding and the closure of the embedding is a disc. The boundary of the disc can be thought of as a relation conjugating the actions of $\pi_1 F$ on the parameter space $\mathbb{RP}^1_L$ and $\mathbb{RP}^1_R$ of the left and right rulings. Such a relation must be a homeomorphism (so the boundary of the discs contains no segment of a line of Q) and ϕ is the unique homeomorphism which conjugates the left and right actions. So the boundary of the closure of the universal cover of F is the boundary $\mathsf{graph}\,\phi$ of $X(\phi)$. Therefore the universal cover of F is equivariantly embedded in $\mathsf{int}\big(H(F(\phi))\big) = \mathsf{int}(H(\mathsf{graph}\,\phi))$. So some neighbourhood of F embeds in $\mathsf{int}\big(H(F(\phi))\big)/\pi_1 F$. If $N(F)$ is a domain of dependence then the development map must define a map taking $N(F)$ to $\mathsf{int}\big(H(F(\phi))\big)/\pi_1 F$. If this were not injective there would be two points in the universal cover of the domain of dependence whose timelike and null geodesics hit the same subset of $\widetilde{F}$, but this is impossible. □

We can reinterpret the boundary components of the convex hull as earthquakes, [14, 40, 41, 20] of which we will now give a fairly self-contained exposition. The simplest earthquakes are fractional Dehn twists, i.e., shears on closed geodesics; a geodesic lamination is a limiting case of a long simple closed geodesic and an earthquake is a shearing motion along a geodesic lamination. Recall that $1+1$-dimensional de Sitter space $\mathbb{RP}^2 - \mathsf{cl}\,\mathbb{H}^2$ can be identified with the set of (unoriented) geodesics: Given $p \in \mathbb{RP}^2 - \mathsf{cl}\,\mathbb{H}^2$ draw the two tangents to the conic S^1_∞, and identify p with the geodesic joining the two points of tangency. The geodesics corresponding to p and q cross, respectively are asymptotic, iff the line joining p and q does not meet S^1_∞, respectively is tangent to S^1_∞, iff p and q are mutually spacelike, respectively mutually null. A geodesic lamination is a partition of a closed subset of $\mathbb{H}^2$ into (disjoint) complete geodesics. Dually it is a closed subset of $\mathbb{RP}^2 - \mathsf{cl}\,\mathbb{H}^2$ containing no mutually spacelike pair of points. A measured lamination (λ, μ) is a closed subset λ of $\mathbb{RP}^2 - \mathsf{cl}\,\mathbb{H}^2$ together with a locally finite positive measure μ such that $\lambda = \mathsf{supp}\,\mu$. Dually it is a geodesic lamination together with a measure defined on transverse arcs. Given a measured lamination (λ, μ), let λ_0 be the union of the leaves of λ which carry atoms of the measure μ. λ_0 consists of countably many leaves, but need not be closed, though if in addition λ is invariant under a Fuchsian group with finite covolume, λ_0 will be closed [19]. There is a metric space $\mathbb{H}^2(\lambda_0)$ in which distances are given by lengths of paths together with a map $p_{\lambda_0} : \mathbb{H}^2(\lambda_0) \to \mathbb{H}^2$ such that if $l \in \lambda_0$ is a geodesic then $p^{-1}_{\lambda_0}(l)$ is isometric to $l \times [0, \mu(l)]$ and p_{λ_0} is the projection onto the factor l. $\mathbb{H}^2(\lambda_0)$ is unique up to a unique isometry, and homeomorphic to $\mathbb{R}^2$. $\mathbb{H}^2(\lambda_0)$ is complete. $\mathbb{H}^2(\lambda_0)$ carries a measured geodesic lamination λ^* without atoms (generalizing the definition in an obvious way) and the push forward of the transverse measure of λ^* equals the transverse measure λ.

A *left earthquake* with *shearing lamination* λ is a relation $f : \mathbb{H}^2 \to \mathbb{H}^2$ such that

i) $f = f^* \circ p^{-1}_{\lambda_0}$ where $f^* : \mathbb{H}^2(\lambda_0) \to \mathbb{H}^2$ is a function,

ii) $f^*|_{p^{-1}_{\lambda_0}(l)}$ maps $p^{-1}_{\lambda_0}(l)$ to a geodesic $f(l)$. f^* is an isometry on each $l \times x$ and each $q \times [0, \mu(l)]$. Given a transverse arc $a : (-\epsilon, \epsilon) \to \mathbb{H}^2$ parametrized by arc length, with $a(0) \in l$ for some geodesic l of the lamination λ, there is an obvious lift $a^* : \big(-\epsilon, \epsilon + \mu(\lambda_0 \cap a(-\epsilon, \epsilon))\big) \to \mathbb{H}^2(\lambda_0)$. It is required that if l is oriented so as to cross $a((-\epsilon, \epsilon))$ from right to left looking in the positive direction along $a((-\epsilon, \epsilon))$, then the lift $a^*|_{[0, \mu(l)]}$ be orientation preserving: $a^*(t)$ moves to the left, as seen looking in the positive direction along $a((-\epsilon, \epsilon))$, as t increases; and

iii) $f^*(x) = x \cdot T(x)$, where $T : \mathbb{H}^2(\lambda_0) \to \mathsf{Iso}^+(\mathbb{H}^2)$ is continuous and, if $a : [0, \epsilon) \to \mathbb{H}^2(\lambda_0)$ is a transverse arc, $T(a(x)) = h(\mu^*(a[0, x]))$ where $\frac{d}{dt}|_{t=0} \frac{h(t)}{h(0)} = t_l$

where t_l is the element of the Lie algebra of $\mathsf{Iso}^+(\mathbb{H}^2)$ which translates l along l with unit speed toward the left, as seen looking in the positive direction along $a((-\epsilon, \epsilon))$.

Now let $\phi : S^1 \to S^1$ be a homeomorphism. Embed S^1 in Q as the intersection C of Q with a spacelike plane P. We identify $q \in Q$ with $(a, b) \in \mathbb{RP}^1_L \times \mathbb{RP}^1_R$ where a, b are the lines through q belonging to the left and right rulings. We identify each of $\mathbb{RP}^1_L, \mathbb{RP}^1_R$ with C by identifying $a \in \mathbb{RP}^1_L$ or $\mathbb{RP}^1_R$ with $a \cap C$. Then $\mathsf{graph}\,(\phi) = \{a \in Q : a = (x, \phi(x))$ for some $x\}$. Now consider the future component $F(\phi)$ of the boundary of the convex hull of $\mathsf{graph}\,(\phi)$. It carries a lamination λ' (i.e., a closed subset partitioned into properly embedded lines): The union of those spacelike geodesics in anti de Sitter space which lie in $F(\phi)$ but do not lie in the interior of the intersection of a spacelike plane with $F(\phi)$. For each geodesic l' in this lamination, with end points a', b' let a, b be the intersections of the geodesic in the left ruling through a, b respectively with the conic C. Let l be the geodesic through a, b. The union of the geodesics l is a geodesic lamination λ on P. Similarly using the right ruling we construct a geodesic lamination λ'' on P. Now given a supporting plane T of $F(\phi)$ there is a unique element $g(T)$ of G_L such that $T \cdot g(T) = P$. $g(T)$ moves each point of $T \cap Q$ along a line of the left ruling to the intersection of that line with P. $g(T)$ is also the unique element of G_L which translates the dual point of T to the dual point of P. Similarly there is a unique element $h(T) \in G_R$. Now we define a left earthquake, which extends continuously to the boundary of $\mathbb{H}^2$ where it equals ϕ. Given a supporting plane T of $F(\phi)$, map $T \cap F(\phi) \cdot g(T)$ to $T \cap F(\phi) \cdot h(T)$ by $(g(T)^{-1}, h(T))$. This defines a relation which is single valued except on sets $T \cap F(\phi) \cdot g(T)$ for which T is not the unique supporting plane at $T \cap F(\phi)$. The lamination λ' carries a bending measure (discussed at length in [44, 20]; there is no essential difference in the case of an indefinite metric) which can be transported to the laminations λ, λ''. One way to describe the bending measure is to associate to a transverse arc the length of the arc of dual points to the supporting planes. To do so one needs to know that the arc of dual points is rectifiable. This follows however from the fact that the direction of the geodesic lamination λ is a Lipschitz continuous function [19].

Proposition 22. *For each orientation preserving homeomorphism $\phi : S^1 \to S^1$ there is a unique left earthquake $E(\phi) : \mathbb{H}^2 \to \mathbb{H}^2$ which extends continuously to $\mathbb{H}^2 \cap S^1$ with value ϕ on S^1. The map $\phi \mapsto E(\phi)$ is equivariant with respect to pre-composition and post-composition with $\mathsf{Iso}^+\mathbb{H}^2$. The shearing lamination and measure are naturally associated to the bending lamination and bending measure of the boundary of the convex hull in anti de Sitter space of the graph of ϕ regarded as a subset of Q. There is also a unique right earthquake obtained from the past boundary of the convex hull.*

Proof. One need only check the definitions, except perhaps for the uniqueness. Given an earthquake, one can construct a locally convex surface with no extreme points with boundary $\mathsf{graph}\,\phi$; this must be the future boundary. The easiest way to check that the future and past boundaries correspond to left and right earthquakes is to draw some examples. (In Figure 6, the earthquake is certainly a left earthquake; the corresponding component of the boundary of the convex hull, which is the union of two spacelike half-spaces, may look at first like the past boundary, but the future

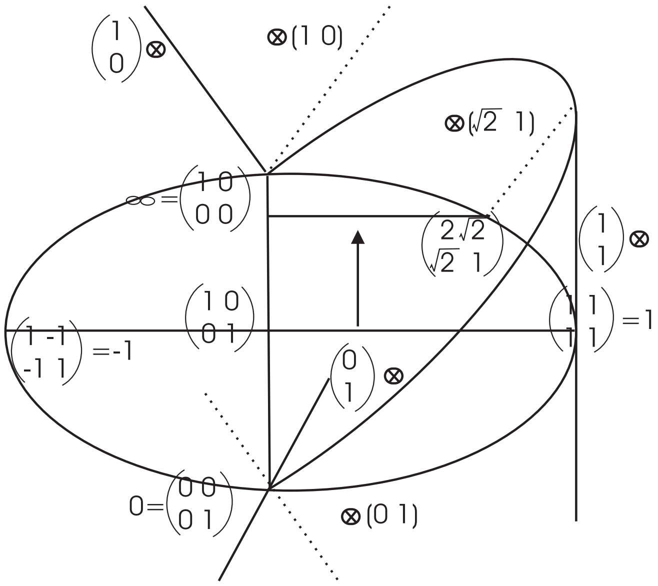

FIGURE 6. A left earthquake

direction along say the subgroup of rotations $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is downwards in the picture.) ☐

Choose P to be the plane $b = c$ in the space of matrices. The intersection of the plane P with Q can be identified with $\mathbb{RP}^1$ by $(a : b : c : d) \mapsto (a : b)$. There is an earthquake $E : \mathbb{H}^2 \to \mathbb{H}^2$ with shearing lamination given by the single geodesic $b = c = 0$ and transverse measure $\log s$. The earthquake can be normalized to be the identity on the half-plane $ab < 0$ and given by $A \mapsto \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$ on the half-plane $ab > 0$. Thus $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} s^2 & s \\ s & 1 \end{pmatrix}$. (For convenience we are using homogeneous coordinates, and matrices in $\mathbf{GL}(2, \mathbb{R})$ rather than in $\mathbf{SL}(2, \mathbb{R})$). The boundary value of the earthquake is the homeomorphism of $\mathbb{RP}^1$ which is the identity on negative reals and multiplication by s on positive reals. The length of the shear along the lamination is $\log s$. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ goes along the line $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \mathbb{R}^2$ to $\begin{pmatrix} s & 1 \\ s & 1 \end{pmatrix}$ and then along the line $\mathbb{R}^2 \otimes \begin{pmatrix} s & 1 \end{pmatrix}$ to $\begin{pmatrix} s^2 & s \\ s & 1 \end{pmatrix} = E(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix})$. The isometry fixing $b = c = 0$ pointwise and taking the plane through $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} s & 1 \\ s & 1 \end{pmatrix}$ to the plane P is given by conjugation by $\begin{pmatrix} s^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}$ and takes $\begin{pmatrix} s & 1 \\ s & 1 \end{pmatrix}$ to $\begin{pmatrix} s & s^{\frac{1}{2}} \\ s^{\frac{1}{2}} & 1 \end{pmatrix}$. Thus after an earthquake with half the transverse measure, there is an isometry with the boundary of the convex hull. As long as the earthquake is along a lamination which is a union of isolated leaves, the same argument is applicable: For after a composition with an isometry one can assume the earthquake is the identity on a region on one side of any given leaf. Any complete spacelike surface in anti de Sitter space which is a boundary component of the convex hull of a nowhere timelike curve in the boundary can be approximated by one with a discrete bending lamination, so in general given a hyperbolic surface S and an earthquake on S determined by a measured geodesic lamination, the quotient of the top of the convex hull in de Sitter space of the graph on the quadric Q of the conjugating homeomorphism determined by the earthquake on S is isometric to the hyperbolic surface obtained by an earthquake with half as much transverse measure.

Proposition 23. *Suppose $E : \mathbb{H}^2 \cup \mathbb{RP}^1 \to \mathbb{H}^2 \cup \mathbb{RP}^1$ is a left earthquake along a lamination λ with transverse measure μ. Then there is an isometry I from the hyperbolic plane to the future component of the convex hull of the graph of the boundary value homeomorphism ϕ of E. Let F be the earthquake with shearing lamination λ and transverse measure $\frac{1}{2}\mu$. Then $I \circ F$ has as boundary value the map which sends $p \in \partial\mathbb{H}^2$ to the corresponding point on the graph of ϕ, i.e., the intersection of the line in the left ruling through p with the graph.*

It would be more natural to formulate proposition 23 in terms of relative hyperbolic structures as in [41]. When an earthquake's shearing lamination (together with its measure) is equivariant with respect to a Fuchsian group π we can consider the induced earthquake on the quotient surface or orbifold; its image lies in the quotient surface or orbifold of a different Fuchsian group π'. Given a homeomorphism of a non-oriented surface, there is a canonical lift to the orientation cover: A point q in the orientation cover is a point p in the non-oriented surface together with a local orientation at p, and the lift maps the q to the image of p together with the image of the local orientation. Thus left and right earthquakes can be defined on nonorientable as well as on orientable surfaces.

Given any measured geodesic lamination $(\lambda', 2\mu)$ one can bend a spacelike geodesic plane along it, obtaining a complete convex surface A and the closure of A will be a disc in $X \cup Q$ with nowhere spacelike boundary. (Here anti de Sitter space has an advantage over hyperbolic space: No matter how large the bending, the map to anti de Sitter space is an isometry.) We regard λ' as lying in A. For each geodesic l' in λ', there are corresponding geodesics l and l'' in $\mathbb{H}^2 = X \cap P$ obtained by left and respectively right translation of the endpoints. It follows, using the previous proposition, that there is an earthquake E defined on $\mathbb{H}^2$ with shearing lamination (λ', μ); up to an isometry the image of λ' is the "lamination" λ'' which is the union of the geodesics l''. λ'' need not be closed. The image of E is a convex open set whose frontier in $\mathbb{H}^2$ is a union of geodesics. Indeed, not all earthquakes have boundary values which are homeomorphisms. The graph in Q of $f : \mathbb{R} \to \mathbb{R}, t \mapsto e^t$ union a single segment of a null line has a convex hull, whose boundary components give earthquakes taking a complete hyperbolic surface with cyclic fundamental group and one cusp to one of the components of the complement of a closed geodesic in a hyperbolic surface with cyclic fundamental group and no cusp.

For another interesting example, suggested by Bowditch, start with the hyperbolic plane considered as the universal cover of the thrice punctured sphere and the preimage of the three geodesics on the thrice punctured sphere which run from one cusp to another, each with unit transverse measure. Bend the plane. The closure $\mathsf{cl}\, A$ of the resulting convex surface meets Q in the limit set of a Fuchsian group which uniformizes a planar surface with three boundary circles, together with sawteeth. That is, there is a spacelike plane P such that $\mathsf{cl}\, A \cap Q \cap P$ is a Cantor set, and for each gap in the Cantor set with endpoints p, q a sawtooth consisting of a segment of a line of the left ruling starting at p and a segment of a line of the right ruling starting at q, each going in the future direction from p or from q. The top surface of the convex hull determines a left earthquake on the interior of the convex core of the planar surface. The leaves of the shearing lamination spiral to the right towards the boundary geodesics. After the earthquake they spiral to the left towards the boundary geodesics. The earthquake with half as much transverse measure turns the surface with geodesic boundary into a thrice punctured sphere, an example of proposition 23. Note that the past boundary of the convex hull in anti de Sitter space contains parts of null planes. One expects the past boundary to determine a right earthquake, defined on the projection to the hyperbolic plane of the spacelike part of the boundary, and since the spacelike part of the past boundary lies in a single plane, the shearing lamination is the zero lamination. That is, the right earthquake is the identity. Given any nowhere spacelike circle in Q there is a corresponding left and right earthquake pair, each defined on the interior of a convex subset of the hyperbolic plane with geodesic boundary. Each conformally null segment of a line in the left ruling in the boundary accounts for one geodesic on the frontier of the range of the left earthquake and one geodesic on the frontier of the domain of the right earthquake.

Let us close with some questions. Given a "quasifuchsian" group acting on anti de Sitter space, the volume of the convex hull and of the domain of dependence are invariants. These volumes are functions on the product of two copies of Teichmüller space. How do they behave? Are they related, perhaps asymptotically, to such invariants of a quasifuchsian group as the volume of the convex hull and the Hausdorff dimension of the limit set? Suppose the hyperbolic structure is specified on the two boundary components of the convex hull; does there exist a unique manifold with the given pair of hyperbolic structures? (In the case of quasifuchsian groups existence follows from the Sullivan-Epstein-Marden theorem [31, 20] but I do not know a proof of uniqueness.) Is a quasifuchsian group determined by the hyperbolic structure on one boundary of the convex hull together with the bending lamination on the other boundary? Is it determined by the two bending laminations? Is it determined uniquely by the conformal structure on the upper surface at infinity together with the hyperbolic structure on the bottom surface of the convex hull? Is it determined by the conformal structure of the upper surface at infinity together with the bending lamination on the bottom surface of the convex hull? Laplace's equation can be solved by complexifying and separating variables, so it looks like the wave equation. Analogously, one can think of the left and right holonomy representations of a closed spacelike surface in anti de Sitter space as analogous to the two conformal structures at infinity of a quasifuchsian group. So there are analogous questions about the representation of the fundamental group of a locally anti de Sitter spacetime which is a neighbourhood of a closed spacelike surface: Is it determined by the two measured laminations, or by the hyperbolic structure on the future boundary of the convex hull together with the measured lamination on the past boundary of the convex hull, or by the left holonomy together with the hyperbolic structure on one of the boundary components of the convex hull?

8. Classification of spacetimes

Recall that a geometric structure on a manifold M is (by definition) complete provided that $M = X/\Gamma$ for some subgroup Γ of G acting properly discontinuously on X. We recall that *i*) Margulis [22, 23], cf. [24, 25] showed the existence of a complete flat 3-dimensional Lorentz manifold with free fundamental group, *ii*) complete closed Lorentz flat manifolds all have virtually polycyclic fundamental group by a theorem of Goldman and Kamishima [28] and complete closed 3-dimensional affine flat manifolds all have polycyclic fundamental groups by a theorem of Fried and Goldman [26] and are classified in [26]; the case of closed flat Lorentz 3-manifolds having been done already in [27], and *iii*) Carrière [5] shows that a compact affine manifold with linear holonomy preserving a Lorentzian structure or more generally of "discompacity one" is complete. For more information see [28] and [46].

Proposition 24. *The linear holonomy of a complete flat spacetime is either solvable or else a discrete subgroup of* $\mathbf{O}(2,1)$.

Proof. Suppose M is a complete oriented orthochronous spacetime. The Lorentzian orthonormal oriented, time-oriented frame bundle of M can be identified with the quotient of the Lorentzian orthonormal oriented time-oriented frame bundle of $\mathbb{R}^{2+1}$ by the holonomy of M, and the Lorentzian orthonormal oriented time-oriented frame bundle of $\mathbb{R}^{2+1}$ can be identified with $\mathbf{ISO}(2,1)$. So $\pi_1 M$ is a discrete subgroup of $\mathbf{ISO}(2,1)$. If $\pi_1 M$ is not solvable and the linear holonomy is not discrete, the linear holonomy must be a dense subgroup of $\mathbf{SO}(2,1)_0$. Furthermore, by Jørgensen's inequality, if every 2-generator subgroup of a finitely generated subgroup H of $\mathbf{SO}(2,1)_0$ is discrete, H is discrete.

Lemma 8. *Suppose A, B generate a linear group which is not virtually solvable and suppose that both A, B have infinite order. Then for all sufficiently large N, A^N and B^N freely generate a free group.*

Proof. See [30] and [33]. □

We may assume the linear holonomy is torsion-free, passing if need be to a finite index subgroup by Selberg's lemma. If the linear holonomy is not discrete, there must be a subgroup $\langle A, B\rangle$ generated by two non-commuting elliptic elements. By Kronecker's theorem on rational approximation, there is a sequence $N_i \to \infty$ such that $A^{N_i}, B^{N_i} \to I$. By the previous lemma, $\langle A^{N_i}, B^{N_i}\rangle$ is a free nonabelian group for all sufficiently large i. So the holonomy contains a free group $\langle S_0, T_0\rangle$ such that $S_0\mathbf{x} = A\mathbf{x} + \mathbf{c}$ and $T_0\mathbf{x} = B\mathbf{x} + \mathbf{d}$. Inductively define $S_{n+1} = [S_n, T_n], T_{n+1} = [T_n, S_n], A_{n+1} = [A_n, B_n], B_{n+1} = [B_n, A_n], S_n\mathbf{x} = A_n\mathbf{x} + \mathbf{c}_n, T_n\mathbf{x} = B_n\mathbf{x} + \mathbf{d}_n$. Choose a norm $\| \quad \|$ on $\mathbb{R}^{2+1}$. There is a constant C such that $\|I - [X,Y]\| + \|I - [Y,X]\| < C(\|X - I\| + \|Y - I\|)^2$ if $\|X - I\| + \|Y - I\|$ is sufficiently small. We have $[S_n, T_n]\mathbf{x} = A_{n+1}\mathbf{x} + A_n^{-1}C_n^{-1}(A - n\mathbf{d}_n - \mathbf{d}_n) + A_n^{-1}C_n^{-1}(\mathbf{c}_n - C_n\mathbf{c}_n)$ and similarly for $[T_n, S_n]$. So if N_i is chosen sufficiently large, S_n, T_n converge to the identity, contradicting the fact that the holonomy is discrete in $\mathbf{ISO}(2,1)$. □

I believe proposition 24 is known.

Proposition 25. *There does not exist a complete flat spacetime M with fundamental group $\pi_1 M \cong \pi_1 S$ where S is a closed surface of negative Euler characteristic.*

Proof. Suppose M exists. Passing to a covering space we may assume M is oriented and orthochronous; then S is an oriented surface. Since M is complete, $\pi_1 M$ injects into $\mathbf{ISO}(2,1)$. Since $\pi_1 M$ has no normal abelian subgroup, the linear holonomy injects $\pi_1 M$ into $\mathbf{SO}(2,1)_0$. By proposition 24, the image is discrete. So M has the holonomy of some standard spacetime.

$\widetilde{M} = \mathbb{R}^{2+1}$ contains the universal covers of M_1 and M_2, where M_1, M_2 respectively are the future, respectively the past, of a future, respectively past directed closed convex spacelike surface. M_1 and M_2 are disjoint by proposition 10. Let M_0 be the quotient of $\mathbb{R}^{2+1} - \mathsf{int}(M_1) \cup \mathsf{int}(M_2)$ by $\pi_1 M$. Since the inclusion of either boundary component of M_0 into M_0 is a homotopy equivalence, M_0 is compact. By the Thurston-Lok theorem, there is a family of flat Lorentz structures on M_0 realizing all small deformations of the holonomy, and in this family a neighbourhood of the original structure consists of spacetimes with strictly convex spacelike boundaries. By the end of the proof of proposition 3, all these structures can be extended to complete flat Lorentz structures on M. But even holding the linear holonomy fixed, there are arbitrarily small deformations of the holonomy which do not act fixed point free on $\mathbb{R}^{2+1}$ and therefore cannot be the deck groups of a covering of M by $\mathbb{R}^{2+1}$. To see this, observe that the measured geodesic laminations supported on unions of simple closed curves are dense in the space of all measured geodesic laminations, and by proposition 14, or directly from the construction of proposition 12, the holonomy groups of such structures cannot act freely on $\mathbb{R}^{2+1}$. □

Margulis [22] conjectures that a complete flat Lorentz manifold with finitely generated free fundamental group has discrete and purely hyperbolic linear holonomy. By proposition 24, the holonomy must be discrete, but the possibility of parabolic elements is not excluded. Propositions 24 and 25 show that a complete flat Lorentz $2+1$ spacetime either has solvable fundamental group, in which case the manifold is homotopy equivalent to a torus or to a Klein bottle or else is a closed manifold, or else its fundamental group is a discrete and non-cocompact subgroup of $\mathbf{O}(2,1)$, in which case, being torsion-free, it is a free group. It seems plausible that a complete flat Lorentz manifold with free fundamental group is diffeomorphic to the interior of a (possibly nonorientable) handlebody.

Propositions 6, 7, 8, 15, 26 motivate the definition of the "space of all classical solutions to Einstein's equation in $2+1$ dimensions on a manifold of the form $S \times (0,1)$ containing S as a closed spacelike surface" as the family of all domains of dependence of a fixed topological type. This family is parametrized by the cotangent space of the Teichmüller space of S and (cf. the remarks following proposition 13) the cotangent space has a natural symplectic structure. (In Witten's paper this symplectic structure is obtained by reduction from an infinite dimensional symplectic manifold.) There is a standard prescription for "quantizing" a cotangent bundle. Given a function on the cotangent space which is either constant or linear on fibers, one can associate an operator on the Hilbert space of square integrable half-densities on Teichmüller space, namely a multiplication operator or a first order differential operator. (A half density is a section of the square root of the bundle of n-forms on a given manifold. For a half density $g = f\sqrt{w}$, $\|g\|^2 = \int |f^2| w$ is invariantly defined on a manifold with no given metric.) Equivalently one can use a Hilbert space of functions on Teichmüller space, relative to the Weil-Petersson volume element. In particular the trace of the linear holonomy of an element of the fundamental group becomes an observable. In general the holonomy of an element is conjugate to the composition of a hyperbolic linear isometry of $\mathbb{R}^{2+1}$ and a translation in the fixed direction by a spacelike vector $\mathbf{v}$. The signed length $l(\mathbf{v})$ of $\mathbf{v}$ is well-defined, using the convention that the future pointing expanding eigenvector, the future pointing contracting eigenvector, and an orthogonal unit spacelike vector $\mathbf{w}$ form a positively oriented triad, and $\mathbf{v} = l(\mathbf{v}) \cdot \mathbf{w}$. The functions $l(\mathbf{v})$ are linear on fibers, so they can be identified with vector fields on Teichmüller space. I believe they are the Hamiltonian vector fields associated to the traces of the linear holonomy of group elements, but leave this question to the reader. The Hilbert space together with these operators is called the solution of quantum gravity in $2+1$ dimensions. In addition, Witten defines remarkable new observables which do not come from the classical observables, i.e., the invariants of the holonomy. (It may seem strange to call a definition the solution of a problem, but until Witten's work it was not clear how to make a reasonable definition.)

The classical uncertainty principle states that a wave function ψ for which the variance of position $\Delta_x^2 = (x^2\psi, \psi) - (x\psi, \psi)^2$ is small has a large uncertainty of momentum $\Delta_p^2 = ((i\partial_x)^2\psi, \psi) - (i\partial_x\psi, \psi)^2$. Because the bottom of the spectrum of the Laplacian on Euclidean space is zero, there are wave functions ψ for which the uncertainty of momentum is arbitrarily small. The Weil-Petersson metric determines a second order operator on Teichmüller space, which measures the uncertainty of the translational part of the holonomy. Because the mapping class group is non-amenable, we expect that the spectrum of the Weil-Petersson Laplacian on Teichmüller space is bounded away from zero. A covering of a compact Riemannian manifold with non-amenable deck group has spectrum bounded away from zero, by Brooks's theorem [45]; the moduli space is noncompact and indeed non-complete, so the theorem does not apply, but still the result is likely to hold. This means that for a wave function in the Hilbert space, there is a lower bound on the uncertainty of the translational part of the holonomy, and unlike the case of Euclidean space the bound is uniform: the uncertainty cannot be made small by making the wave function very widely spread over Teichmüller space. We leave these questions to the reader.

In proposition 26 we give an extension of Carrière's theorem to flat Lorentz spacetimes with spacelike boundary, following his proof as closely as possible for the reader's convenience. (Our proof is terse, so the reader may wish to read Carrière's proof first.) In proposition 27 we will consider anti de Sitter manifolds with spacelike boundary.

Definition 5. *Given a flat Lorentzian manifold (or more generally a Lorentzian manifold of constant curvature) N without boundary, let $E_x \subset T_xN$ denote the set of vectors v such that the geodesic flow F_tv of v is defined for all $t \in [0,1]$. E_x is the domain of the exponential.*

Proposition 26. *Suppose M is a compact flat $2+1$-dimensional orthochronous Lorentzian manifold with spacelike boundary. Then if ∂M is empty M is complete. Otherwise M is a product $\partial_0 M \times [0,1]$ with each slice $\partial_0 M \times \{t\}$ spacelike.*

Proof. First we enlarge M. If a component $\partial_{1i}M$ of the future boundary is future directed there is a future complete manifold F_iM with past boundary $\partial_{1i}M$. Adjoin each manifold F_iM to M along $\partial_{1i}M$. If a component $\partial_{1i}M$ is past directed, there is a manifold G_iM such that the frontier of the domain of dependence of the universal cover of $\partial_{1i}M$ is locally convex surface B_{1i} with null supporting planes, and G_iM is the quotient of the region between B_{1i} and the universal cover of $\partial_{1i}M$. Adjoin each manifold G_iM to M along $\partial_{1i}M$. Similarly enlarge M along the past boundary. If there are toral boundary components adjoin their entire futures or pasts (according as they are on the future or past boundary.) Now we have an open manifold M'.

Let $(\widehat{M'},\ p : \widehat{M'} \to M')$ be the holonomy cover of M'. $\widehat{M'}$ contains the holonomy cover $\widehat{M}$ of M as a closed submanifold with boundary. Given $x \in \widehat{M'}$ we will show that E_x, the domain of the exponential at x, is convex. First we consider F_x, the intersection of E_x with the future pointing timelike and null vectors. Suppose $y, z \in F_x$ and there is some $w \notin E_x$ on the line segment $[y, z]$. Then there exists s_0 maximal such that $[s_0y, s_0z] \subset F_x$ for $0 \le s \le s_0$. Consider the convex hull of the complement of F_x in the cone bounded by the rays through y, z. Because a dense subset of the supporting planes of a convex set meet the convex set in an extreme point, it is possible to replace y, z by nearby points so that all of the closed triangle $\triangle 0yz$ except one point w on $[y, z]$ lies in F_x.

Now consider the geodesic ray $r = \mathsf{Exp}[0, w)$; it is the projection to $\widehat{M'}$ of the geodesic flow F_tw of w for $0 \le t < 1$. If r leaves the submanifold $\widehat{M}$ of $\widehat{M'}$ through a boundary component of $\widehat{M}$ which is the past (respectively future) boundary of a future (respectively past) complete component of $\widehat{M'} - \widehat{M}$ then r extends to a complete ray contradicting the fact that $\mathsf{Exp}[0, w)$ is a maximal ray. If r leaves the submanifold $\widehat{M}$ and enters one of the preimages of a manifold G_iM, then the preimage in E_x of the strictly convex surface B_{1i} contains w and is strictly convex toward x (that is, an arc of the preimage of B_{1i} together with rays from its endpoints to 0 enclose a convex bounded set in the tangent space $T_x\widehat{M'}$.) This is impossible because all of $[y, z]$ except w is in E_x. So the ray r remains in $\widehat{M}$. Consider the projection $p(r)$ in M. Since $p(r)$ is timelike, incomplete and maximal and ∂M is compact and spacelike there is a neighbourhood U of ∂M such that at all sufficiently large times r, $p(r)$ is not in U. M is compact so $p(r)$ is recurrent: There is an ellipsoid ϵ_0 in the interior of M such that $p(r)$ infinitely often enters the ellipsoid $p(\frac{\epsilon_0}{2})$ where $\frac{\epsilon_0}{2}$ is the ellipsoid obtained from ϵ_0 by the affine map which fixes the center of ϵ_0 and conjugates a translation by v to a translation by $v/2$ for any $v \in \mathbb{R}^{2+1}$. We suppose that ϵ_0 is small enough that $p(\epsilon_0)$ is embedded. Let $\gamma_i \in \pi_1 M$ be the elements defined by the successive entries of $p(r)$ into $\frac{1}{2}p(\epsilon_0)$. Now consider the preimages in the triangle $\triangle 0yz$ by the exponential map of the ellipsoids $\gamma_i \cdot \epsilon_0$. After passing to a subsequence of $\{\gamma_i\}$, the preimages of the ellipsoids must converge in the Hausdorff topology (cf. [5]) to a line segment or a line through w, and then the exponential of a line segment from 0 to a point near w on the segment $[y, z]$ must pass through $p(\epsilon_0)$ infinitely many times, which is impossible for a compact segment. So the domain F_x is convex and similarly for the intersection P_x of E_x with the past pointing timelike and null vectors. The convergence of the subsequence of ellipsoids to a line or line segment is the crucial idea in Carrière's proof.

Now suppose E_x is not convex. As before we obtain y, z in E_x and a single point w on $[y, z]$ which is not in E_x. If y, z are relatively timelike, we obtain a contradiction: The ray $p([0, w))$ cannot reach a preimage of a boundary component B_{1i} because then $[y, z]$ would map to a ray crossing B_{1i}. Now assume y and z are relatively spacelike. Suppose there are points w_n arbitrarily near w on $[0, w)$ such that $p(\mathsf{Exp}(w_n))$ lies in M. Then (because each component of ∂M is locally convex or locally concave) there is a slightly smaller compact manifold $M_0 \subset \mathsf{int}M$ such that $p(\mathsf{Exp}[0,w))$ is recurrent in M_0. Then there are ellipsoids $p(\gamma_n \cdot \frac{\epsilon_0}{2})$ which cross the ray $p(\mathsf{Exp}[0,w))$ and (after taking a subsequence) the ellipsoids converge. The limit is necessarily the union of all the null lines through an interval of $[y,z]$ containing w. (Otherwise some segment in the open triangle $\triangle 0yz$ would cross infinitely many ellipsoids.) We obtain a contradiction because a compact segment $[0,w']$ where w' is near w on $[y,z]$ crosses infinitely many ellipsoids. So the ray $p(\mathsf{Exp}[0,w])$ eventually leaves $M \subset M'$. If it left through a concave boundary component – that is, a future, respectively a past boundary component of M which is the past, respectively future boundary of a future, respectively past complete end of M then it eventually reenters M. But if it leaves through a convex boundary component we obtain a contradiction because the universal cover of the domain of dependence of a spacelike surface is convex, by proposition 11. So E_x is convex.

By a result of Koszul (proposition 1.3.2 of [5]), $\widehat{M'}$ is E_x, regarded as an open subset of $\mathbb{R}^{2+1}$. In the case that M is closed, Carrière concludes that the frontier of $\widehat{M}$ consists of $0, 1$ or 2 hyperplanes and deduces that in fact $\widehat{M} = \mathbb{R}^{2+1}$. Now suppose $\widehat{M}$ has at least two future boundary components $\partial_{1i}\widehat{M}, \partial_{1j}\widehat{M}$. At most one of these covers the past boundary in M' of a future complete submanifold of M'. Otherwise $\widehat{M'}$ would not be embedded in $\mathbb{R}^{2+1}$. Also at most one of $\partial_{1i}\widehat{M}, \partial_{1j}\widehat{M}$ is convex. Otherwise a timelike linear functional would have two local maxima on the closure of $\widehat{M'}$ in $\mathbb{R}^{2+1}$. If one, say $\partial_{1i}\widehat{M}$, is convex and another covers the past boundary of a future complete manifold then it follows that the convex manifold $\widehat{M'}$ is all of $\mathbb{R}^{2+1}$ contradicting the assumption that $\partial_{1i}\widehat{M}$ is convex.

We conclude that if M has a boundary component $\partial_i M$ then $\pi_1 \partial_i M$ has index one in $\pi_1 M$. If follows that (up to time reversal) either $\widehat{M}$ has one future boundary component which is convex and one which is concave, or else M has two boundary components each of which is a torus and $\widehat{M'} = \mathbb{R}^{2+1}$. In the first case, for any timelike direction $\mathbf{v}$ there is a family of planes $P_t(\mathbf{v}) = \{\mathbf{x} \cdot \mathbf{v} = ta(\mathbf{v}) + (1-t)b(\mathbf{v})\}$ such that $P_0\mathbf{v}$ and $P_1\mathbf{v}$ are support planes of the future and past boundaries respectively. For each t, the envelope of all the planes P_t is a spacelike surface, and this surface defines an explicit product structure on M with spacelike slices. If the past or future boundary of M is a torus whose fundamental group has index one in $\pi_1 M$ then the theorem holds using proposition 8. $\square$

Let X denote anti de Sitter space and $\widetilde{X}$ the universal cover of anti de Sitter space. Suppose M is a compact oriented orthochronous anti de Sitter manifold with spacelike boundary. We consider M as an $(\mathsf{Iso}\widetilde{X}, \widetilde{X})$-manifold, so the holonomy cover $\widehat{M}$ develops into $\widetilde{X}$.

Proposition 27. *a) Given a point $x \in \widehat{M}$, the exponential is a diffeomorphism from the intersection of the domain E_x of the exponential with any spacelike plane in the tangent space $T_x\widehat{M}$ to a convex set in a spacelike plane in $\widehat{M}$. b) If M is a compact oriented orthochronous anti de Sitter manifold with spacelike boundary, then either M embeds in a domain of dependence or else M is a closed anti de Sitter manifold whose universal cover is a domain in X with locally convex frontier.*

Proof. The boundary components of M must have negative Euler characteristic by proposition 21. Also, using proposition 21 M can be thickened to a manifold with locally strictly convex boundary. Let P be a spacelike plane in $\widetilde{X}$, and let U be a connected component of $\mathsf{dev}^{-1}P$. Suppose $x \in U$. Let T_xP be the plane in $T_x\widehat{M}$ tangent to U. P is a hyperbolic plane, so the exponential map is injective on T_xP. The proof of proposition 26 extends to show that E_x meets T_xP in a convex set. Since $\widehat{M}$ has locally convex boundary, the subset $\mathsf{Exp}(E_x \cap T_xP) \cap U$ is convex. Choosing a subgroup $\mathbb{R} = \widetilde{\mathbf{SO}(2)} \subset G_L$ defines a time function $t : \widetilde{X} \to \mathbb{R}$ such that the action of $\mathbb{R}$ permutes the planes $P_s = t^{-1}(s), s \in \mathbb{R}$. So $\widehat{M}$ is foliated by the preimages by dev of the planes P_s. Now we will show that the preimage of a spacelike plane P is connected.

Suppose $\mathsf{dev}^{-1}P_0$ is not connected, but contains disjoint components U, V. There is some maximal interval $[a,b]$ containing 0 such that $\mathsf{dev}^{-1}t^{-1}[a,b]$ contains two distinct components U', V' containing U, V. One but not both of a, b may be infinite; suppose b is finite. Then $\mathsf{dev}(U') \cap P_b$ and $\mathsf{dev}(V') \cap P_b$ are disjoint open convex sets, perhaps with disjoint closures, but for $c > b$ sufficiently near b there are points $p \in U' \cap \mathsf{dev}^{-1}P_b$, $q \in V' \cap \mathsf{dev}^{-1}P_b$ and x such that $\mathsf{dev}(x) \in P_c$ and such that $\widehat{M}$ contains a triangle T, i.e., a set such that $\mathsf{dev}(T)$ is the closed triangle $\triangle\mathsf{dev}(p)\mathsf{dev}(q)\mathsf{dev}(x)$ minus a closed interval in the segment $\mathsf{dev}(p)\mathsf{dev}(q)$. The set $\triangle\mathsf{dev}(p)\mathsf{dev}(q)\mathsf{dev}(x)$ lies in a timelike plane. As in proposition 26 we obtain that $\widehat{M}$ contains a triangle $\triangle p'q'x$ with one point missing from the closed interval $p'q'$. By the argument of proposition 26, this is a contradiction.

So $\widehat{M}$ is foliated by open convex spacelike totally geodesic surfaces and is embedded in $\widetilde{X}$. It follows that if ∂M is nonempty then $\widehat{M}$ has only one past component and only one future component. Therefore the inclusion of either boundary component of M induces an isomorphism of fundamental groups. Furthermore, $\widehat{M}$ has locally convex frontier in $\widetilde{X}$ and at every point of the frontier there is a null supporting plane. In this case either $\widehat{M}$ is contained in a domain of dependence or else $\widehat{M}$ contains the entire region bounded by the past boundary component $\partial_0\widehat{M}$ of $\widehat{M}$ and a translate $T \cdot \partial_0\widehat{M}$ where T generates the deck group of the covering of X by $\widetilde{X}$ and moves points to points in their futures. But then there would be a domain of dependence $H(\partial_0\widehat{M})$ such that $\pi_1\partial_0\widehat{M}$ acted properly discontinuously on a region properly containing the entire future component of the frontier of the domain of dependence $H(\partial_0\widehat{M})$ and this is impossible. Indeed, the action on the future component of the frontier of the domain of dependence is topologically equivalent to the action of $\pi_1 M'$ on the frontier of the universal cover of M', for some flat Lorentz manifold M' which is a domain of dependence. So we may assume M is closed. We consider M as a $(\mathsf{Iso}\widetilde{X}, \widetilde{X})$-manifold, so the development maps the holonomy cover $\widehat{M}$ to $\widetilde{X}$. We have shown that the development embeds $\widehat{M}$ as a simply connected open subset of $\widetilde{X}$, since there is a submersion to $\mathbb{R}$ whose fibers are connected and simply connected. So we can regard $\pi_1 M$ as a subgroup of $\mathsf{Iso}\widetilde{X}$. First we consider the case that $\pi_1 M$ meets the center $\mathbb{Z}$ of $\mathsf{Iso}\widetilde{X}$ in the subgroup $n\mathbb{Z}, n > 0$. In this case we can consider M as an $(\mathsf{Iso}X_n, X_n)$-manifold where X_n is the n-fold cyclic cover of $\mathbf{PSL}_2\mathbb{R}$; considered as such the holonomy cover (which we still denoted by $\widehat{M}$) of M develops to X_n and this map is an embedding such that the inclusion of the image is a homotopy equivalence with X_n. We will identify $\widehat{M}$ with its image. Fix $p \in \widehat{M}$. There is a smooth timelike loop $c : S^1 = \mathbb{R}/\mathbb{Z} \to \widehat{M}$ which represents the generator of $\pi_1 X_n$. For each t there is a timelike geodesic arc d_t, parametrized by arc length, in X_n joining 0 to $c(t)$. Choose c so that for $t \neq 0 \in S^1$, $c(t)$ is not one of the n translates of $p = c(0)$ by the center $\mathbb{Z}/n$ of $\mathsf{Iso}X_n$. Then d_t depends continuously on t. The set of t for which d_t lies in $\widehat{M}$ is open. It is also closed. Indeed, suppose that for $0 \leq t \leq t_0$, d_t lies in $\widehat{M}$. Let L be the length of d_{t_0}. The set of parameter values s such that $d_{t_0}(s)$ is in $\widehat{M}$ is either all of $[0, L]$ or else there is some $l < L$ such that $d_{t_0}([0, L))$ lies in $\widehat{M}$ but $d_{t_0}(L)$ does not. Then after a small change in $p = c(0)$ and $d_{t_0}(L)$ we obtain, as in proposition 26, a triangle $\triangle c(0)d_{t_0}(L)x$ in X_n lying in a timelike plane such that all of the triangle except one point on the side $c(0)d_{t_0}(L)$ lies in $\widehat{M}$. By Carrière's recurrence argument (cf. proposition 26) this is impossible. So $\widehat{M}$ contains a closed timelike curve through p, say with unit tangent vector $\mathbf{v}$ at p. The set of unit timelike vectors which are tangent to closed curves in $\widehat{M}$ is open, because all timelike geodesics in X_n are closed and $\widehat{M}$ is open in X_n. It is also closed, by another application of Carrière's argument. So $\widehat{M}$ contains the entire hyperbolic plane $\mathbb{H}^2(p)$ which is one component of the preimage in X_n of the dual plane in $\mathbf{PSL}_2\mathbb{R} = X_1$ to the image of the point p in X_1. Given any point of $\mathbb{H}^2(p)$, one (namely the normal) and therefore all of the unit timelike vectors through that point are tangent to a closed geodesic in $\widehat{M}$. We conclude that M is complete. M has "finite level" in the terminology of [53].

Now suppose that $\pi_1 M$ has trivial intersection with the center $\mathbb{Z}$ of $\mathsf{Iso}\widetilde{X}$. $\widehat{M}$ is an open subset of $\widetilde{X}$. Suppose for the moment that $\widehat{M}$ is a proper open subset. The frontier of $\widehat{M}$ is locally convex and defined by null supporting planes. Furthermore, $\widehat{M}$ is disjoint from its translates by the center of $\mathsf{Iso}\widetilde{X}$. Otherwise the frontier of $\widehat{M}$ would meet the interior of a translate of $\widehat{M}$. But $\pi_1 M$ acts properly discontinuously on the interior of $\widehat{M}$ and all of its translates, but not at any point of the frontier, because any point of the frontier is the endpoint of a ray which has recurrent projection to M and therefore is the accumulation point of an orbit. So we can consider M as an $(\mathsf{Iso}X, X)$-manifold which is the quotient of a domain by a cocompact subgroup. The past and future boundaries of this domain are well-defined. By the arguments of proposition 20 the frontier of either boundary component $\widehat{M}$ in $\mathbb{RP}^3$ meets the quadric at infinity in a nowhere timelike topological circle. In fact this is the entire intersection of the frontier with the quadric at infinity. For if the circles obtained from the past and future boundary components were not equal, the quadric would contain some open set which lay between them. Every point in the open set would be a point of the frontier. Otherwise there would be a geodesic ray leaving $\widehat{M}$ without passing through either the past or future boundary. So if the frontier of the past and future boundary components were not equal, the frontier would meet the quadric in a closed set invariant under $\pi_1 M$ and containing an open set. This closed set would necessarily equal all of the quadric at infinity, which contradicts the hypothesis that $\widehat{M}$ is a proper open subset of X. So the frontier of $\widehat{M}$ in $\mathbb{RP}^3$ meets the quadric at infinity in a nowhere timelike topological circle, which we call the limit circle. This contains at most countably many straight segments. If say the left holonomy had nontrivial kernel, any element in the kernel would fix all of the limit circle except possibly some open segments. It would therefore fix set-wise three distinct null rays of the right ruling on the quadric and therefore also have trivial right holonomy. If follows that neither the left nor the right holonomies can have nontrivial kernel. Suppose the limit circle contained segments of lines. Then one, say the left, holonomy leaves invariant a collection of open intervals in

$\mathbb{RP}^1$, from which it follows that the left holonomy has free image. But since the left holonomy has trivial kernel, this implies that M has free fundamental group. But M is a closed aspherical manifold of dimension 3, so this is impossible. So the limit circle contains no segments of lines. It therefore conjugates the left and right holonomies. Since $\mathbb{RP}^1_L$ and $\mathbb{RP}^1_R$ can be identified by the conjugating homeomorphism ϕ it follows from the discreteness of $\pi_1 M$ in $G_L \times G_R$, and the fact that G_L can be identified with the space of distinct triples of points in $\mathbb{RP}^1_L$ that $\pi_1 M$ is discrete regarded as a subset of G_L. But this implies that $\pi_1 M$ is a Fuchsian group which contradicts the fact that $\pi_1 M$ is the fundamental group of a closed aspherical 3-manifold. We conclude that M is complete.

□

Kulkarni and Raymond [53] showed that a closed 3-manifold with a complete anti de Sitter structure is necessarily a Seifert manifold. See also [54] for more information on closed 3-manifolds with complete anti de Sitter structure. [54] shows that given a complete Lorentzian manifold of the form $\mathbf{PSL}_2\mathbb{R}/\Gamma$ where Γ is a cocompact subgroup acting by right translation, the deformations with left holonomy nontrivial but lying in an abelian subgroup are still complete. By the Thurston-Lok holonomy theorem, there are also deformations of the Lorentzian structure corresponding to any small change in the holonomy; in general the left holonomy will be irreducible after such a change. Goldman conjectured in [54] that these manifolds would still be complete; this is established by proposition 27. It seems likely that by similar arguments one could prove there does not exist a closed de Sitter manifold and that a compact de Sitter manifold with spacelike boundary is a product in which all the slices are spacelike, if one knew that a de Sitter manifold which was a tubular neighbourhood of a closed spacelike surface was associated to a projective structure.

In the case of flat and anti de Sitter spacetimes containing a closed spacelike surface, we showed that a neighbourhood of the universal cover of the surface embeds in the model space. The identity component of the isometry group of the model space acts simply transitively on the oriented orthochronous Lorentzian orthonormal frame bundle of the model space. Since the holonomy acts properly discontinuously on the neighbourhood of the surface, and therefore also on the restriction of the frame bundle to this neighbourhood, the holonomy of the surface must be discrete in the isometry group. This argument together with proposition 24 is an alternative, in the case of a flat spacetime, to the use of Goldman's theorem, and works in $n+1$ dimensions. Proposition 4 also generalizes. One can deduce that a $3+1$-dimensional spacetime which is a small neighbourhood of a closed spacelike hypersurface and has irreducible linear holonomy is a deformation, by a 1-cocycle with values in $\mathbb{R}^{3+1}$, of the quotient of the positive (or negative) light cone by a group acting cocompactly on the hyperbolic 3-space of unit timelike directions.

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Mathematical Sciences Research Institute, Institute for Advanced Study, Institut des Hautes Études Scientifiques, and University of California at Los Angeles