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Flat Conformal Structures and Causality in de Sitter Manifolds

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DEDICATION

For Gail.

TABLE OF CONTENTS

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ABSTRACT OF THE DISSERTATION

Flat Conformal Structures and Causality in de Sitter Manifolds

by

Kevin Patrick Scannell

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Professor Geoffrey Mess, Chair

Given a compact n-manifold Σ with a flat conformal structure, there is a canonical procedure for constructing an associated $(n + 1)$-dimensional de Sitter spacetime homeomorphic to $\Sigma \times (0, \infty)$; we call these *standard de Sitter spacetimes*. Our main theorem is a classification of compact de Sitter manifolds, complementing results of G. Mess in the flat and anti-de Sitter cases. The first part of the classification asserts that every de Sitter spacetime which is a small regular neighborhood of a compact spacelike hypersurface isometrically embeds in a standard de Sitter spacetime. This fact is used to obtain our second main result, which states that a compact $(2 + 1)$-dimensional de Sitter spacetime with non-empty spacelike boundary is homeomorphic to a product $\Sigma \times [0, 1]$.

CHAPTER 1

Introduction

By a *spacetime*, we shall mean a connected, oriented, time-oriented Lorentz manifold. A *de Sitter* (resp. *flat*, *anti-de Sitter*) *spacetime* is a spacetime of constant positive (resp. zero, negative) curvature. In [2], Carrière uses an elegant geometric argument to show that a compact flat spacetime without boundary is geodesically complete. This result has been extended by Mess [35] and Klingler [26] to the anti-de Sitter and de Sitter cases respectively. A *spacetime-bordism* is a compact spacetime with non-empty spacelike boundary, which we can view as a bordism between the past and future boundary components. The general approach to classifying spacetime-bordisms of constant curvature (see Witten [47], [48]) is to first classify those spacetimes which are small regular neighborhoods of closed spacelike hypersurfaces, and then to show that an arbitrary spacetime-bordism is actually homeomorphic to a product $\Sigma \times [0,1]$ (typically with spacelike slices $\Sigma \times \{t\}$). Mess [35] has successfully carried out this program in the three-dimensional flat and anti-de Sitter cases (see Appendix). In this paper we finish the classification by providing a complete solution in the de Sitter case.

Suppose Σ is a compact n-manifold without boundary, equipped with a flat conformal structure. Thurston has given a construction by means of which one can "thicken" a developing map $dev : \tilde{\Sigma} \to \mathbb{S}^n$ of the flat conformal structure to obtain an equivariant immersion $D : \tilde{\Sigma} \times (0, \infty) \to \mathbb{H}^{n+1}$. In dimension two, this was used by Thurston to parameterize $\mathbb{C}P^1$-structures on Σ by the space of measured geodesic laminations on Σ (these arise as "bending laminations" on the frontier of the image of D). The projective dual of this construction gives an equivariant immersion of $\tilde{\Sigma} \times (0, \infty)$ into $(n+1)$-dimensional de Sitter space, inducing a de Sitter metric on $\Sigma \times (0, \infty)$; the spacetimes obtained in this way are the *standard de Sitter spacetimes*, constructed in detail in §6. The theorem below gives the first part of the two-step classification program outlined above, and verifies a conjecture of Mess [35].

Theorem 1.1 *Every de Sitter spacetime which is a small regular neighborhood of a compact spacelike hypersurface isometrically embeds in a standard de Sitter spacetime.*

Recall that a spacetime $\mathcal{M}$ is a *domain of dependence* if it contains a global Cauchy hypersurface, i.e. a closed spacelike hypersurface Σ such that every inextendible causal curve in $\mathcal{M}$ meets Σ exactly once (see §3). A theorem of Geroch [12] shows that the interior of a domain of dependence is homeomorphic to a product $\Sigma \times \mathbb{R}$ in such a way that each slice $\Sigma \times \{t\}$ is a global Cauchy hypersurface, and thus the second part of the classification program (in dimension three) will follow from our other main result:

Theorem 1.2 *Every three-dimensional de Sitter spacetime-bordism is a domain of dependence, with the exception of those standard de Sitter spacetimes arising from closed two-dimensional Hopf manifolds.*

In the process of proving Theorem 1.2, we will obtain necessary and sufficient conditions for the appearance of a non-trivial causal horizon in a standard de Sitter spacetime:

Theorem 1.3 *Suppose* Σ *is a closed, orientable surface with a* $\mathbb{C}P^1$*-structure and let* $\mathcal{M} \approx \Sigma \times (0, \infty)$ *be the associated standard de Sitter spacetime. Then* $\mathcal{M}$ *is a domain of dependence, and embeds in a strictly larger de Sitter spacetime if and only if* Σ *contains a codimension-zero Hopf manifold.*

CHAPTER 2

Geometric Structures and Deformation Spaces

Good references for the material presented in this chapter are [1], [16], and [45]. Suppose G is a Lie group which acts faithfully, transitively, and analytically on a manifold X. Let M be a connected $C^{0,1}$ manifold, possibly with boundary, with a fixed basepoint $m_0 \in M$. By convention, the universal cover a space will always be indicated by the addition of a tilde, so $\tilde{M}$ denotes the universal cover of M.

A *based* (G, X)*-structure* on M is a pair (f, ϕ) consisting of a $C^{0,1}$ local embedding $f : \tilde{M} \to X$, and a homomorphism $\phi : \pi_1(M, m_0) \to G$ satisfying:

$$f(\gamma \cdot x) = \phi(\gamma) \cdot f(x), \tag{2.1}$$

for all $\gamma \in \pi_1(M, m_0)$ and all $x \in \tilde{M}$ (we say f is ϕ*-equivariant*). The homomorphism ϕ is called the *holonomy representation* of the based (G, X)-structure, and f is called the *developing map*. Let $\mathcal{D}_{(G,X)}(M)$ denote the set of based (G, X)-structures on M, identifying pairs which differ by the action of a diffeomorphism $g : (M, m_0) \to (M, m_0)$ isotopic to the identity rel m_0. Projection onto the second component induces a well-defined map $\mathcal{D}_{(G,X)}(M) \to \mathrm{Hom}(\pi_1(M, m_0), G)$. The Thurston-Lok holonomy theorem asserts that when M is compact and of the same dimension as X, this map is a local homeomorphism with respect to the natural topologies [16],[32],[45, §5.1]. The *deformation space* $\mathcal{T}_{(G,X)}(M)$ *of* (G, X)*-structures on* M is defined to be the quotient of $\mathcal{D}_{(G,X)}(M)$ under conjugation by G. A (G, X)*-manifold* is a pair consisting of a connected, $C^{0,1}$ manifold M and a point in $\mathcal{T}_{(G,X)}(M)$. We will habitually abuse terminology by referring to an element of $\mathcal{T}_{(G,X)}(M)$ by a representative based (G, X)-structure.

Note that a based (G, X)-structure can only exist if the dimension of M is less than or equal to the dimension of X. When these dimensions are equal, a based (G, X)-structure on M defines a (G, X)*-structure* in the classical sense; that is, a maximal atlas of charts $\{\phi_\alpha : U_\alpha \to X\}$ such that each overlap map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to X$ is given by the restriction of the action of an element of G on each connected component of $\phi_\alpha(U_\alpha \cap U_\beta)$. Conversely, given a (G, X)-structure in this sense, the standard analytic continuation construction recovers a holonomy representation and an equivariant developing map, unique up to conjugation by an element of G.

Our primary examples of geometric structures will come from the constant curvature Riemannian and Lorentzian model spaces. Fix integers $0 \leq k \leq n$ with $n \geq 2$, and define $\mathbb{R}^n_k$ to be the space $\mathbb{R}^n$ equipped with the signature $(n - k, k)$

inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle = -\sum_{i=1}^{k} v_i w_i + \sum_{j=k+1}^{n} v_j w_j. \tag{2.2}$$

When $k = 1$, we call $\mathbb{R}^n_1$ *(flat) Minkowski space.* Recall that a vector $\mathbf{v} \in \mathbb{R}^n_1$ is said to be:

- *spacelike* if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$;
- *null* or *lightlike* if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$;
- *timelike* if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$;
- *causal* if $\langle \mathbf{v}, \mathbf{v} \rangle \leq 0$.

Define:

$$\mathbb{S}^n_1 = \{\mathbf{v} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{v}, \mathbf{v} \rangle = 1\} \tag{2.3}$$

and

$$\mathbb{H}^n_1 = \{\mathbf{v} \in \mathbb{R}^{n+1}_2 \mid \langle \mathbf{v}, \mathbf{v} \rangle = -1\}. \tag{2.4}$$

$\mathbb{S}^n_1$ and $\mathbb{H}^n_1$ are the models for n-dimensional *de Sitter space* and *anti-de Sitter space* and inherit Lorentz metrics of constant curvature $+1$ and -1 respectively. Note that $\mathbb{S}^n_1$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$, and admits a natural conformal compactification $\overline{\mathbb{S}^n_1} \approx \mathbb{S}^{n-1} \times [0, 1]$ by $(n-1)$-spheres $\partial^-_\infty \mathbb{S}^n_1$ and $\partial^+_\infty \mathbb{S}^n_1$ at past and future infinity respectively.

An alternative model of de Sitter space is constructed by means of the natural projection $\varpi : \mathbb{R}^{n+1}_1 \setminus \{0\} \to \mathbb{R}P^n$. Define $(\mathbb{H}^n)^*$ to be the image in $\mathbb{R}P^n$ of the spacelike vectors of $\mathbb{R}^{n+1}_1$; we call $(\mathbb{H}^n)^*$ the *projective model* of de Sitter space. Note that the restriction of ϖ to $\mathbb{S}^n_1$ induces a double covering of $(\mathbb{H}^n)^*$. Recall that the image in $\mathbb{R}P^n$ of the timelike vectors of $\mathbb{R}^{n+1}_1$ is the usual projective (Klein) model of n-dimensional hyperbolic space $\mathbb{H}^n$, with the projectivized null vectors corresponding to the sphere at infinity $\partial_\infty \mathbb{H}^n$; thus $\partial_\infty \mathbb{H}^n$ simultaneously compactifies $\mathbb{H}^n$ and $(\mathbb{H}^n)^*$. The advantage of this model is that we may exploit the projective duality between k-planes in $\mathbb{H}^n$ and $(n-k-1)$-planes in $(\mathbb{H}^n)^*$ to transfer certain standard constructions from hyperbolic space to de Sitter space (this also motivates our unusual notation $(\mathbb{H}^n)^*$).

In light of the above discussion, we will be considering families of (G, X)-manifolds with $G = SO_0(n, 1)$, the identity component of $O(n, 1)$. This group is simultaneously isomorphic to the group $Isom^+(\mathbb{H}^n)$ of orientation-preserving isometries of $\mathbb{H}^n$, the group $M\ddot{o}b^+(\mathbb{S}^{n-1})$ of orientation-preserving Möbius transformations of $\mathbb{S}^{n-1}$, and the group $Isom^+_\uparrow(\mathbb{S}^n_1)$ of orientation-preserving, orthochronous isometries of $\mathbb{S}^n_1$. Corresponding to these three identifications, we have the following examples of (G, X)-structures on an orientable n-manifold:

- A *hyperbolic structure* is an $(SO_0(n,1), \mathbb{H}^n)$-structure. The existence of a hyperbolic structure on an orientable n-manifold is equivalent to the existence of a Riemannian metric of constant negative curvature.

- A *flat conformal structure* is an $(SO_0(n+1,1), \mathbb{S}^n)$ structure. In dimension two this is simply the classical notion of a *projective* or $\mathbb{C}P^1$*-structure* on a Riemann surface. For $n \geq 3$, recall Liouville's Theorem which states that a conformal homeomorphism of domains in $\mathbb{S}^n$ is the restriction of a Möbius transformation. It follows that a flat conformal structure is equivalent to a (locally) conformally flat Riemannian metric [27], [34].

- A *de Sitter structure* is an $(SO_0(n,1), \mathbb{S}^n_1)$-structure. With this terminology, a de Sitter spacetime is exactly the same as a n-manifold with a de Sitter structure.

We will use the abbreviations $\mathbb{H}^n(M)$, $\mathcal{C}(M)$, and $\mathbb{S}^n_1(M)$ for the respective deformation spaces $\mathcal{T}_{(G,X)}(M)$ of hyperbolic, flat conformal, and de Sitter structures on an n-manifold M.

CHAPTER 3

Elementary Causality

A C^1 submanifold of a Lorentz manifold is *spacelike* (resp. *null, timelike, causal*) if all its tangent vectors are spacelike (resp. null, timelike, causal). Of course, only one-dimensional submanifolds can be null, timelike, or causal; we therefore define a submanifold to be *Lorentzian* (resp. *degenerate*) if the induced metric is everywhere Lorentzian (resp. degenerate). Recall that we have assumed all spacetimes to be time-oriented, so any causal curve is either *future-pointing* or *past-pointing*.

Let $\mathcal{M}$ be a spacetime, and consider a point $x \in \mathcal{M}$. Define $I^+(x)$ to be the set of points $p \in \mathcal{M}$ such that there exists a non-trivial past-pointing timelike curve from p to x (when $\mathcal{M} = \mathbb{S}^n_1$, this definition works equally well for points $x \in \partial^-_\infty \mathbb{S}^n_1$). If Σ is a subset of $\mathcal{M}$, let $I^+(\Sigma) = \cup_{x\in\Sigma} I^+(x)$. The set $I^+(\Sigma)$ is clearly open, and is called the *chronological future* of Σ in $\mathcal{M}$. The *chronological past* $I^-(x)$ is defined by replacing "past-pointing" with "future-pointing" in the definition of $I^+(x)$; similarly, define $I^-(\Sigma) = \cup_{x\in\Sigma} I^-(x)$. The *future domain of dependence* $D^+(\Sigma)$ is defined to be the set of points $p \in \mathcal{M}$ such that every inextendible past-pointing causal curve starting at p intersects Σ. The *future Cauchy horizon* is given by

$$H^+(\Sigma) = \overline{D^+(\Sigma)} \setminus D^+(\Sigma). \tag{3.1}$$

The sets $D^-(\Sigma)$ and $H^-(\Sigma)$ are defined analogously. We say Σ is *achronal* (resp. *acausal*) if no timelike (resp. causal) curve intersects Σ more than once. A *(topological) hypersurface* Σ in a spacetime is a subset which can be covered by charts $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ which map the pairs $(U_\alpha, U_\alpha \cap \Sigma)$ homeomorphically to $(\mathbb{R}^n, \mathbb{R}^{n-1})$. We remark that a closed achronal hypersurface Σ is necessarily *edgeless*; i.e. for each $x \in \Sigma$, there is a neighborhood U of x such that every timelike curve in U joining points of $I^+(x)$ and $I^-(x)$ must meet Σ. Finally, Σ is called a *global Cauchy hypersurface* for $\mathcal{M}$ if it is a closed, spacelike, acausal hypersurface, and $\mathcal{M} = D^+(\Sigma) \cup D^-(\Sigma)$; when such a hypersurface exists $\mathcal{M}$ is said to be a *domain of dependence*.

Lemma 3.1 *[41, Ch. 14, Lemma 43] If Σ is a closed acausal hypersurface, then $D^+(\Sigma) \cup D^-(\Sigma)$ is open.* ■

If we are given a closed, spacelike, acausal hypersurface $\Sigma \subset \mathcal{M}$, then to show Σ is a global Cauchy hypersurface for $\mathcal{M}$ it suffices by this lemma to show that $H^+(\Sigma) = H^-(\Sigma) = \emptyset$. Showing that the Cauchy horizons vanish is facilitated by the following elementary characterization of $H^+(\Sigma)$, which may be assembled from the standard references (e.g. chapter 6 of [20] and chapter 14 of [41]).

Lemma 3.2 *Suppose Σ is a closed acausal hypersurface. Then if $H^+(\Sigma)$ is non-empty, it is a closed achronal $C^{0,1}$ hypersurface disjoint from Σ. Furthermore, a point x is in $H^+(\Sigma)$ if and only if the following two conditions hold:*

- *every inextendible past-pointing timelike curve starting at x intersects Σ;*
- *there exists an inextendible past-pointing null geodesic ray starting at x which lies entirely within $H^+(\Sigma)$.*

■

Here and in what follows, our results are stated for the future Cauchy horizon $H^+(\Sigma)$, the statements for $H^-(\Sigma)$ being completely analogous. The null geodesic rays given by Lemma 3.2 are called the *null generators* of $H^+(\Sigma)$.

We will now specialize the discussion of causality to the special case of hypersurfaces in constant curvature spacetimes. A *spacelike de Sitter hypersurface* is a compact, oriented, smooth n-manifold Σ without boundary, equipped with a based $(SO_0(n+1,1), \mathbb{S}_1^{n+1})$-structure (f, ϕ) such that f is a spacelike immersion; it follows that Σ inherits a well-defined complete Riemannian metric. Similarly there are notions of *spacelike flat hypersurface* and *spacelike anti-de Sitter hypersurface*. Given a spacelike de Sitter hypersurface Σ, let $\mathcal{M} = \Sigma \times (0, \infty)$ and define $\mathfrak{D}(\Sigma) \subseteq \mathbb{S}_1^{n+1}(\mathcal{M})$ to be the set of all de Sitter structures on $\mathcal{M}$ such that there exists an isometric embedding of Σ as a global Cauchy hypersurface for $\mathcal{M}$. The set $\mathfrak{D}(\Sigma)$ is non-empty and partially ordered by inclusion, hence by Zorn's lemma there is a maximal element $\mathcal{M}_{max}(\Sigma)$. From the existence of the developing map, one sees that the "germ of extensions" is unique; i.e. for any two elements $\mathcal{M}_1, \mathcal{M}_2 \in \mathfrak{D}(\Sigma)$ there exist $\mathcal{M}_3 \in \mathfrak{D}(\Sigma)$ and isometric embeddings of $\mathcal{M}_3$ in both $\mathcal{M}_1$ and $\mathcal{M}_2$. Using this fact and an argument of Choquet-Bruhat and Geroch ([4],[20, Ch. 7]), one can show that every element of $\mathfrak{D}(\Sigma)$ isometrically embeds in $\mathcal{M}_{max}(\Sigma)$ (compare [1, §1.6]). We identify Σ with its image in $\mathcal{M}_{max}(\Sigma)$. Again, a similar construction works for the flat and anti-de Sitter cases. The spacetime $\mathcal{M}_{max}(\Sigma)$ is called the *maximal domain of dependence* for Σ; it is homeomorphic to $\Sigma \times \mathbb{R}$ using the result of Geroch cited in the introduction.

Proposition 3.3 *Suppose Σ is a simply-connected spacelike de Sitter (resp. flat, anti-de Sitter) hypersurface. Then $\mathcal{M}_{max}(\Sigma)$ is either future complete or else embeds in a de Sitter (resp. flat, anti-de Sitter) spacetime in which the following conditions hold:*

1. *$H^+(\Sigma)$ is non-empty;*
2. *Every null generator of $H^+(\Sigma)$ is past complete;*
3. *Every null generator is either future complete or contains a future endpoint.*

Proof: Let $X = \mathbb{S}^{n+1}_1, \mathbb{R}^{n+1}_1$, or $\mathbb{H}^{n+1}_1$ as appropriate, and let $dev : \mathcal{M}_{max}(\Sigma) \to X$ denote the developing map (here we have used the fact that $\mathcal{M}_{max}(\Sigma) \approx \Sigma \times \mathbb{R}$ is simply-connected). Define an equivalence relation on the set of inextendible future-pointing timelike curves in $\mathcal{M}_{max}(\Sigma)$ by declaring $\lambda_1 \sim \lambda_2$ if and only if $I^-(\lambda_1) = I^-(\lambda_2)$. Let H denote the set of equivalence classes of future-incomplete inextendible timelike curves in $\mathcal{M}_{max}(\Sigma)$, and let $\mathcal{M}' = \mathcal{M}_{max}(\Sigma) \cup H$. There is an obvious extension of dev to $\mathcal{M}'$ which sends an equivalence class of curves to the common future endpoint of their developing images. We define a topology on $\mathcal{M}'$ for which a typical basis element $\mathcal{U}_{\lambda,x}$ containing the equivalence class $[\lambda] \in H$ is obtained by choosing a point x lying on λ and setting

$$\mathcal{U}_{\lambda,x} = I^+(x) \cup \{[\lambda'] \in H \mid \text{ some representative } \lambda' \subset I^+(x)\}. \tag{3.2}$$

The extension of dev to $\mathcal{M}'$ is continuous with respect to this topology. In this way, we induce causal relations between points in $\mathcal{M}'$; because $\mathcal{M}_{max}(\Sigma)$ is the maximal domain of dependence, $H^+(\Sigma) = H$. The argument for Proposition 3.2 which shows that the Cauchy horizon is a closed achronal hypersurface works here as well, and we conclude that $\mathcal{M}'$ is a constant curvature spacetime with boundary $H^+(\Sigma)$. For the remainder of the proof we will work in the enlarged manifold $\mathcal{M}'$.

Consider an arbitrary point $x \in H^+(\Sigma)$ lying on a null generator λ. Let $\{\beta_j\}$ be a sequence of inextendible past-pointing timelike curves starting at x and approaching λ. Suppose λ is past incomplete, and let $p \in X$ be the past endpoint of $dev(\lambda)$. It follows that only finitely many of the curves $dev(\beta_j)$ enter $I^-(p)$, or else we could construct a timelike curve back in $\mathcal{M}'$ corresponding to the missing endpoint of λ. Thus infinitely many of the $dev(\beta_j)$ meet $dev(\Sigma)$ before reaching $I^-(p)$; this contradicts the completeness of Σ.

Consider a sequence of points x_j which lie on a null generator for $H^+(\Sigma)$ such that x_j is to the past of x_{j+1}, and suppose $dev(x_j) \to p$. Take a past-pointing timelike segment from each point x_j, so that the endpoints form a timelike-separated sequence $\{z_j\}$ in $\mathcal{M}_{max}(\Sigma)$ and $dev(z_j) \to p$. These points can be joined by a future-pointing timelike curve whose equivalence class is the limit of the x_j. We conclude that the null generators are closed sets and (3) follows. ∎

When $\mathcal{M}_{max}(\Sigma)$ fails to be future complete, the spacetime given by Proposition 3.3 will be denoted $\overline{\mathcal{M}}_{max}(\Sigma)$. If $\mathcal{M}_{max}(\Sigma)$ is future complete, we simply set $\overline{\mathcal{M}}_{max}(\Sigma) = \mathcal{M}_{max}(\Sigma)$.

It should be remarked that much of the discussion above can be simplified in the flat and anti-de Sitter cases. It is quite easy to prove, for instance, that an edgeless, spacelike immersion of an $(n-1)$-manifold into $\mathbb{R}^n_1$ or $\mathbb{H}^n_1$ is in fact an achronal embedding. This is far from true for spacelike de Sitter hypersurfaces however, and presents the main difficulty in studying the global structure of de Sitter spacetimes.

Next, we recall some terminology which will arise repeatedly in our discussion of the de Sitter/hyperbolic duality. A subset of $\mathbb{S}^n$ is an *open round ball* if it is the

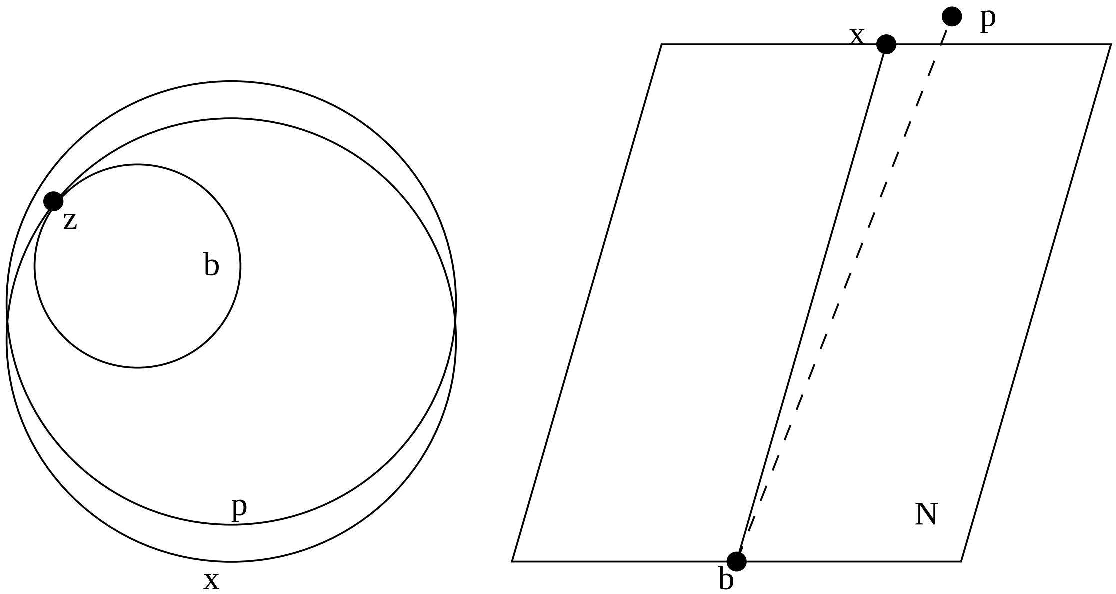

Figure 3.1: The figure on the left illustrates the proof of Proposition 3.4 in the case $n = 3$ on the two-sphere at past infinity $\partial^-_\infty \mathbb{S}^3_1$; the figure on the right is the corresponding picture inside $\mathbb{S}^3_1$.

image of an open hemisphere under a Möbius transformation. A *round* $(n-1)$-*sphere* in $\mathbb{S}^n$ is the boundary of an open round ball. Recall that a point in $\mathbb{S}^n_1$ projects to $(\mathbb{H}^n)^*$ and is therefore dual to a hyperplane in $\mathbb{H}^n$; this in turn defines a round $(n-2)$-sphere and two complementary open round balls on the sphere at infinity. The following trivial piece of information provides a key element in our later development.

Proposition 3.4 *Let λ be a past complete null ray in $\mathbb{S}^n_1$, $\mathbb{R}^n_1$, or $\mathbb{H}^n_1$. Then there is a unique degenerate hyperplane N containing λ and $I^+(\lambda) = I^+(N)$.*

Proof: Consider first the case when $\lambda \subset \mathbb{R}^n_1$; without loss of generality we may assume λ is a line through the origin in the direction of some vector $\mathbf{n} \in \mathbb{R}^n_1$. Let $N = \mathbf{n}^\perp$; that is, the subspace

$$\mathbf{n}^\perp = \{\mathbf{v} \in \mathbb{R}^n_1 \mid \langle \mathbf{n}, \mathbf{v} \rangle = 0\}. \tag{3.3}$$

It follows easily that N is the unique degenerate hyperplane containing λ. Also note that N is foliated by the collection of null lines parallel to λ. Clearly $I^+(\lambda) \subseteq I^+(N)$; for the converse, consider a point $\mathbf{w} \in I^+(N)$. The past-pointing null cone $\partial I^-(\mathbf{w})$ intersects the hyperplane N in a paraboloid whose axis of symmetry is parallel with λ, hence each of the null lines foliating N intersects $I^-(\mathbf{w})$ (in particular, λ does). Thus $\mathbf{w} \in I^+(\lambda)$.

In de Sitter space $\mathbb{S}^n_1$, the degenerate k-planes are precisely the intersections with $\mathbb{S}^n_1$ of the degenerate $(k+1)$-planes through the origin in $\mathbb{R}^{n+1}_1$. Using this remark, the result for de Sitter space follows easily. A similar proof works for anti-de Sitter space.

Alternatively, we can give an argument for $\mathbb{S}^n_1$ which emphasizes the duality with hyperbolic space. Take a point $p \in I^+(N) \subset \mathbb{S}^n_1$ and let x be the future starting point of λ. The ray λ meets past infinity at a point $z \in \partial^-_\infty \mathbb{S}^n_1$; the unique degenerate hyperplane N containing λ is the union of all null rays converging to z. The points p and x correspond to round $(n-2)$-spheres p^* and x^* such that z lies on x^*, while p^* misses z. For simplicity, given a round $(n-2)$-sphere $C^* \subset \partial^-_\infty \mathbb{S}^n_1$ dual to a specific choice of $C \in \mathbb{S}^n_1$, we will refer to the component of $\partial^-_\infty \mathbb{S}^n_1 \setminus C^*$ corresponding to $I^-(C)$ as the "interior" of C^*; in particular z is in the interior of p^*. We see immediately that there exists a open round ball b^* tangent to x^* at z which is contained entirely within the interior of both x^* and p^*. Interpreted in $\mathbb{S}^n_1$, this means that there exists a point b lying on λ (since λ is past complete) which can be reached from p by a past-pointing timelike curve, hence $p \in I^+(\lambda)$. ∎

CHAPTER 4

The Canonical Stratification and Metric

Throughout this chapter, we let Σ denote a compact, connected n-dimensional manifold without boundary. Let $D_\infty : \tilde{\Sigma} \to \mathbb{S}^n$ be a developing map and ϕ a holonomy representation for a flat conformal structure on Σ. A construction originally due to Thurston (unpublished, see [23] however) and extended by Kulkarni-Pinkall [28, 29] produces a canonical decomposition of Σ with respect to this structure. This technique will be used to construct families of hyperbolic and de Sitter structures on $\Sigma \times (0, \infty)$ parameterized by $\mathcal{C}(\Sigma)$.

We begin by using D_∞ to pull back the usual metric on $\mathbb{S}^n$ to a metric on $\tilde{\Sigma}$, and considering the metric space completion $\overline{\tilde{\Sigma}}$ of $\tilde{\Sigma}$. There is a unique continuous extension $\overline{D}_\infty : \overline{\tilde{\Sigma}} \to \mathbb{S}^n$ of D_∞. A subset $U \subset \tilde{\Sigma}$ is an *open round ball* if D_∞ maps U homeomorphically onto an open round ball in $\mathbb{S}^n$. Given an open round ball U in $\tilde{\Sigma}$, the closure $\overline{U}$ in $\overline{\tilde{\Sigma}}$ maps homeomorphically to a closed round ball in $\mathbb{S}^n$, hence $\overline{U}$ is conformally equivalent to compactified hyperbolic space $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$. We may therefore transfer the usual notion of "hyperbolic convex hull" to $\overline{U}$; let $U_\infty = \overline{U} \setminus \tilde{\Sigma}$ and let $C(U)$ denote the intersection of U and the convex hull of U_∞ in $\overline{U}$ (note that $C(U) = \emptyset$ if and only if U_∞ has fewer than two points).

Proposition 4.1 *Exactly one of the following holds:*

1. $\tilde{\Sigma} \cong \mathbb{S}^n$ *with the obvious flat conformal structure;*
2. $\tilde{\Sigma} \cong \mathbb{E}^n = \mathbb{S}^n \setminus \{\infty\}$*;*
3. *For every* $p \in \tilde{\Sigma}$*, there exists a unique open round ball* U_p *such that* $p \in C(U_p)$*.*

Proof: Fix $p \in \tilde{\Sigma}$, and let W_p be the union of all open round balls containing p (this set is non-empty because D_∞ is a local diffeomorphism). One checks easily that the restriction of D_∞ to W_p is injective, because D_∞ is injective on the union of any two open round balls meeting in a "spherical lens". Let $F = \mathbb{S}^n \setminus D_\infty(W_p)$; this set is the intersection of closed round balls in $\mathbb{S}^n$, and is therefore a closed convex set. Suppose F has fewer than two points. Then W_p is conformally equivalent to either $\mathbb{S}^n$ or $\mathbb{E}^n$, and if the dimension of $\tilde{\Sigma}$ is at least two it follows that $\tilde{\Sigma} = \overline{W_p} \cong \mathbb{S}^n$ or $\mathbb{E}^n$ (in the one-dimensional case, we obtain the same conclusion without necessarily having $\tilde{\Sigma} = \overline{W_p}$). We shall assume therefore that F has at least two points and without loss of generality that $D_\infty(p) = \infty \in \mathbb{S}^n$, so we can view F as a subset of $\mathbb{E}^n = \mathbb{S}^n \setminus \{\infty\}$. Hence there exists a unique closed round ball B of least radius containing F; since D_∞ is injective on W_p, the set $U_p = D_\infty^{-1}(\mathbb{S}^n \setminus B)$ is an open round ball in $\tilde{\Sigma}$. We claim $p \in C(U_p)$.

By tracing through the definitions, we have that $\overline{D}_\infty((U_p)_\infty) = F \cap \partial B$ and so $p \in C(U_p)$ if and only if $D_\infty(p)$ is in the convex hull of $F \cap \partial B$ (taken in the complement of B). By inversion in ∂B, this in turn is equivalent to the Euclidean center of B lying in the convex hull of $F \cap \partial B$ (taken in B). If this failed to hold however, one could construct a closed round ball of lesser radius containing F.

Uniqueness of U_p is clear, the cogent remark being that for any pair of open round balls U_1 and U_2 in $\mathbb{S}^n$, the convex hull of $\partial U_1 \setminus U_2$ in U_1 and the convex hull of $\partial U_2 \setminus U_1$ in U_2 must be disjoint. ■

The flat conformal structure is said to be of *elliptic type*, *parabolic type*, or *hyperbolic type*, depending on whether (1), (2), or (3) holds in the statement of Proposition 4.1. In the case of hyperbolic type, the decomposition $\tilde{\Sigma} = \bigcup_{p \in \tilde{\Sigma}} C(U_p)$ is called the *canonical stratification* of $\tilde{\Sigma}$; each $C(U_p)$ is a called a *stratum*. The set of strata is written $\mathcal{S}$; we will provide $\mathcal{S}$ with a canonical metric space structure in a later chapter. Note finally that this decomposition is equivariant with respect to the action of $\pi_1(\Sigma)$ on $\tilde{\Sigma}$, and so there is an induced stratification of Σ.

An open round ball U in $\tilde{\Sigma}$ has a well-defined Riemannian metric pulled back from the Poincaré metric on $D_\infty(U)$ and denoted g_U. Define a metric on all of $\tilde{\Sigma}$ by $g = g_{U_p}\big|_{C(U_p)}$; it is shown in [29] that g is a complete $C^{1,1}$ Riemannian metric, with almost-everywhere-defined sectional curvatures in the interval $[-1, 1]$. Each stratum in the canonical stratification is totally geodesic with respect to g. Again, the construction of g is equivariant and therefore induces a Riemannian metric on Σ with the same properties. When the dimension of Σ is two, this metric can be used to obtain a complete hyperbolic structure on Σ, and the space of strata yields a measured geodesic lamination. This case is discussed in more detail in §9 below.

CHAPTER 5

Examples of Flat Conformal Structures

There is a large body of literature concerning flat conformal structures, particularly in the classical case of $\mathbb{C}P^1$-structures on surfaces. We will now present some examples indicating the wide variety of behavior one can expect.

Example 1: Consider $\mathbb{S}^n$ with its trivial flat conformal structure, and suppose $\mathbb{S}^k \subset \mathbb{S}^n$ is a round k-sphere for some $k = 0, \dots, n-1$. Then $\mathbb{S}^n \setminus \mathbb{S}^k$ also (trivially) admits a flat conformal structure, and when equipped with the canonical metric constructed in §4, is isometric to $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$. In particular, when $k = 0$ (and passing to the universal cover if in addition $n = 2$), we obtain the n-dimensional *simply-connected Hopf manifold.* If Σ is an n-manifold with a flat conformal structure, and the lifted flat conformal structure on $\tilde{\Sigma}$ is induced by an embedding of $\tilde{\Sigma}$ into the simply-connected n-dimensional Hopf manifold, we will say Σ is a *Hopf manifold.* The classical examples come from the quotient of an invariant subset of $\mathbb{S}^n \setminus \{0, \infty\}$ under the action of a loxodromic element in $SO_0(n+1, 1)$ fixing 0 and ∞. When $n = 2$ and there is no rotational component of the loxodromic, these spaces are known as *θ-annuli*, where θ refers to the width of the invariant region of $\widetilde{\mathbb{S}^2 \setminus \{0, \infty\}}$ in question. A closed manifold with a flat conformal structure of hyperbolic type and abelian holonomy must be a Hopf manifold [34].

Another important class of examples arises when $k = n-2$; these are the so-called *Mercator manifolds.* Mercator manifolds typically arise after performing Thurston's "bending deformation" along a totally geodesic hypersurface in a closed hyperbolic n-manifold (see [29]). Note that when $n = 2$, this notion coincides precisely with the notion of a Hopf manifold.

Example 2: Given a Fuchsian representation $\phi : \pi_1(\Sigma) \to SO_0(3, 1)$, there is a natural $\mathbb{C}P^1$-structure on Σ with developing map D_∞ carrying $\tilde{\Sigma}$ homeomorphically to a ϕ-invariant disk in $\mathbb{S}^2$. Goldman [15] has shown in general that the $\mathbb{C}P^1$-structures with Fuchsian holonomy correspond under Thurston's parameterization (§9) to "integer points" of $\mathfrak{ML}(\Sigma)$, i.e. to finite collections of simple closed geodesics with transverse measures which are integer multiples of 2π. Such structures give the simplest examples of geometric structures for which the developing map is not a covering of its image. Examples of this kind first appeared (in various contexts) in papers by Maskit [33], Hejhal [22], Faltings [8], and Sullivan-Thurston [44].

Example 3: Suppose now that $\phi : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ is a quasi-Fuchsian representation, and that D_∞ maps $\tilde{\Sigma}$ homeomorphically to an invariant topological disk Δ. In this case, the measured lamination associated to the $\mathbb{C}P^1$-structure resides on a boundary component of the convex hull of the quasi-Fuchsian limit set. Note that the Poincaré metric on $\Delta/\phi(\pi_1(\Sigma))$ is not necessarily the same as the canonical metric coming from the flat conformal structure. Sullivan has shown, however, that these metrics are K-quasi-isometric for a constant K independent of ϕ ([6], [43]). This remarkable fact is intimately related to the existence of hyperbolic structures on 3-manifolds fibering over $\mathbb{S}^1$ [46].

Example 4: In addition to Thurston's parameterization of $\mathbb{C}P^1$-structures, there is a well-known analytic parameterization by holomorphic quadratic differentials. Suppose Σ is a closed orientable surface with a fixed uniformization $\Sigma \cong \mathbb{H}^2/\Gamma$. Given a $\mathbb{C}P^1$-structure (D, ϕ) on Σ, let ψ denote the Schwarzian derivative of D:

$$\psi = (D''/D')' - \frac{1}{2}(D''/D')^2. \tag{5.1}$$

The ϕ-equivariance of D is equivalent to the property that for each $z \in \mathbb{H}^2$ and each $\gamma \in \Gamma$

$$\psi(z) = (\psi \circ \gamma(z))(\gamma'(z))^2, \tag{5.2}$$

i.e. $\psi(z)dz^2$ represents a holomorphic quadratic differential on Σ. Conversely, every holomorphic quadratic differential arises in this way, and hence we obtain a parameterization of the space of $\mathbb{C}P^1$-structures on a fixed Riemann surface. See [19] and the references cited therein concerning this approach.

Example 5: The holonomy representation need not be as nicely behaved as in the previous examples. In fact, it has been shown that any homomorphism $\phi : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ which lifts to $SL(2, \mathbb{C})$ and has non-elementary image is realized as the holonomy representation of some $\mathbb{C}P^1$-structure, and therefore, as we shall see, of some standard de Sitter spacetime as well. Gallo originally announced a proof of this fact in [10]; the details have only recently appeared in [25] and [11].

Example 6: The question of which higher-dimensional manifolds admit flat conformal structures remains open despite significant progress over the last twenty years, particularly in dimension three. Thurston's hyperbolization theorem [36] is the most powerful positive result, showing that a vast collection of compact three-manifolds admit hyperbolic structures and therefore flat conformal structures. It is conjectured in [18] that a circle bundle M over a closed surface Σ admits a flat conformal structure if and only if the Euler number $e(M)$ satisfies:

$$|e(M)| \leq |\chi(\Sigma)| \tag{5.3}$$

This has been verified in the case $\chi(\Sigma) = 0$ by Goldman [13], while the first known examples with $e(M) \neq 0$ appeared simultaneously in [18] and in the work of M. Kapovich (surveyed in [24]). Finally we remark that Kulkarni-Pinkall [28] have given methods for gluing flat conformal structures along boundary components under certain hypotheses, greatly expanding the body of known examples.

CHAPTER 6

Standard de Sitter Spacetimes

A C^1 path $\alpha : [0,1] \to \mathcal{M}$ in a spacetime $\mathcal{M}$ has length defined by

$$L(\alpha) = \int_\alpha |\langle \dot{\alpha}, \dot{\alpha} \rangle|^{\frac{1}{2}} \tag{6.1}$$

An *admissible spacelike partition* (resp. *timelike*, *causal*) for a continuous path $\alpha : [0,1] \to \mathcal{M}$ is a finite partition $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ such that for every $j \in 0, \ldots, k-1$, the points $\alpha(t_j)$ and $\alpha(t_{j+1})$ can be joined by a spacelike (resp. timelike, causal) segment $[\alpha(t_j), \alpha(t_{j+1})]$ in a convex normal neighborhood of $\alpha(t_j)$. A continuous path $\alpha : [0,1] \to \mathcal{M}$ is said to be *spacelike* (resp. *timelike*, *causal*) if it has arbitrarily fine admissible spacelike (resp. timelike, causal) partitions (this generalizes the notion for differentiable paths given in §3). When this is the case, we can define the length of such a path as an infimum over all admissible partitions of the appropriate type:

$$L(\alpha) = \inf\{L[\alpha(t_0), \alpha(t_1)] + \cdots + L[\alpha(t_{k-1}), \alpha(t_k)]\}. \tag{6.2}$$

Note that it makes sense to define the length using the infimum versus the supremum, because L satisfies the *reverse* triangle inequality, so refining a partition reduces the sum in (6.2).

The *timelike separation* of points $x, y \in \mathcal{M}$ is defined to be:

$$\tau(x, y) = \sup\{L(\alpha) \mid \alpha \text{ is a causal curve joining } x \text{ and } y\} \tag{6.3}$$

If there are no causal curves joining x and y, then we set $\tau(x, y) = 0$. One verifies easily that τ is symmetric and also satisfies the reverse triangle inequality.

When $\mathcal{M} = \mathbb{S}^n_1$, we can reinterpret these length measurements in terms of the dual hyperplanes in $\mathbb{H}^n$. Points which are spacelike-separated correspond to intersecting hyperplanes, and the length of the shortest segment between them is precisely the dihedral angle between the hyperplanes. Similarly, points which are timelike-separated correspond to disjoint, non-asymptotic hyperplanes and τ is simply the minimum hyperbolic distance between the hyperplanes (recall this is the length of the unique common perpendicular).

For each $x \in \partial^+\mathbb{S}^n_1$ fix a future-pointing timelike geodesic c with arclength parameter which converges to x and define the *timelike horofunction* $\overline{\tau}_x : I^-(x) \to (0, \infty)$ by:

$$\overline{\tau}_x(y) = \lim_{t\to\infty} \tau(y, c(t)) - t. \tag{6.4}$$

Using the reverse triangle inequality, the expression on the right-hand side increases in t and is bounded above, so the limit exists. The function so-defined is independent of the choice of c up to an additive constant.

Now let Σ denote a compact n-manifold without boundary, with a fixed flat conformal structure $(D_\infty, \phi) \in \mathcal{C}(\Sigma)$ of hyperbolic type, and space of strata $\mathcal{S}$ in $\tilde{\Sigma}$. We start the construction of the standard de Sitter spacetimes by defining a map $D_0^* : \mathcal{S} \to (\mathbb{H}^{n+1})^*$. Recall that each stratum $s \in \mathcal{S}$ corresponds to a unique open round ball $U \subset \tilde{\Sigma}$; the set $\partial D_\infty(U)$ bounds a hyperplane in $\mathbb{H}^{n+1}$ which determines the desired point $D_0^*(s)$ in $(\mathbb{H}^{n+1})^*$. Clearly nearby pairs of points in the image of this map are spacelike-separated, for if not, the open round ball corresponding to one of the points would be contained in the interior of the other (with perhaps one common boundary point) – this is impossible if each open round ball defines a non-empty stratum. It follows that any path in $\mathcal{S}$ maps to a continuous spacelike path in de Sitter space, and therefore has an induced length. This defines a metric space structure on $\mathcal{S}$.

Next note that there is a canonical map from $\tilde{\Sigma}$ to $\mathcal{S}$, given by $p \mapsto C(U_p)$; the composition with D_0^* defines a map of $\tilde{\Sigma}$ into $(\mathbb{H}^{n+1})^*$, which by abuse of notation we again denote D_0^*. Define a map $D^* : \tilde{\Sigma} \times (0, \infty) \to (\mathbb{H}^{n+1})^*$ by sending (p, t) to the point on the unique timelike ray from $D_0^*(p)$ to $D_\infty(p)$ satisfying $\tau(D_0^*(p), D^*(p, t)) = t$. This map can be lifted to $\mathbb{S}_1^{n+1}$ in such a way that as $t \to \infty$ the image approaches past infinity; we also write D^* for the lifted map.

If Σ is of parabolic type and $x \in \mathbb{S}^n$ is the point missed by D_∞, then we can define $D^* : \tilde{\Sigma} \times (0, \infty) \to \mathbb{S}_1^{n+1}$ by sending (p, t) to the point on the unique timelike ray from $x \in \partial_\infty^+ \mathbb{S}_1^{n+1}$ to $D_\infty(p) \in \partial_\infty^- \mathbb{S}_1^{n+1}$ satisfying $\overline{\tau}_x(D^*(p, t)) = -\log t$ (since τ is only well-defined up to an additive constant, D^* in this case is well-defined up to a multiplicative rescaling of $(0, \infty)$).

Finally, if Σ is of elliptic type, then we use the homeomorphism of $\mathbb{S}_1^{n+1}$ with $\mathbb{S}^n \times \mathbb{R}$ (coming from its embedding in $\mathbb{R}_1^{n+2}$) to define D^*; as above we simply rescale $(0, \infty)$ by $t \mapsto -\log t$.

By yet another abuse of notation, the composition $\pi_1(\Sigma \times (0, \infty)) \cong \pi_1(\Sigma) \xrightarrow{\phi} SO_0(n, 1)$ will also be denoted ϕ.

Proposition 6.1 *The pair (D^*, ϕ) defines a de Sitter structure on $\Sigma \times (0, \infty)$ which is past complete. For every $t \in (0, \infty)$, the slice $\Sigma \times \{t\}$ is a global Cauchy hypersurface.*

Proof: We will first show that D^* is a ϕ-equivariant C^1 immersion. This is clear in the elliptic and parabolic cases; we may therefore restrict our attention to the case that Σ is of hyperbolic type. It has already been remarked that the canonical stratification is equivariant, so given $\gamma \in \pi_1(\Sigma)$ and $p \in \tilde{\Sigma}$, we have $C(U_{\gamma \cdot p}) = \gamma \cdot C(U_p)$. The ϕ-equivariance of D_∞ then implies that D_0^* and hence D^* are also ϕ-equivariant. The differentiability of D^* can be proven by adapting the dual argument of Bowditch found in [6].

The proof that each slice is a global Cauchy hypersurface requires no further mention of the specific de Sitter structure involved. A slice $\Sigma \times \{t_0\}$ is a closed spacelike hypersurface by construction, and is clearly acausal since it is spacelike and separates $\Sigma \times (0, \infty)$. By Lemma 3.1, $D^+(\Sigma \times \{t_0\}) \cup D^-(\Sigma \times \{t_0\})$ is open, so it suffices to show that this set is also closed. We will show $H^+(\Sigma \times \{t_0\}) = H^-(\Sigma \times \{t_0\}) = \emptyset$. In light of Lemma 3.2, let β be a null geodesic and define $L = \{t \in (0, \infty) \mid \beta \cap (\Sigma \times \{t\}) \neq \emptyset\}$. This set is non-empty and clearly open because each slice is spacelike and β is a null curve. Suppose $\{t_j\}$ is a sequence of points in L converging to some value $t \in (0, \infty)$; then by compactness of the slices, there is some point $z \in \Sigma \times \{t\}$ such that β enters arbitrarily small convex normal neighborhoods of z. Again using the fact that the slices are spacelike, this forces β to intersect $\Sigma \times \{t\}$, and so L is closed. We conclude $L = (0, \infty)$, finishing the proof. ■

Proposition 6.1 provides a well-defined map $\Omega^+ : \mathcal{C}(\Sigma) \to \mathbb{S}^{n+1}_1(\Sigma \times (0, \infty))$. By reversing the time-orientations in each case (e.g. in the hyperbolic case, choosing the other possible lift of D^* from $(\mathbb{H}^{n+1})^*$ to $\mathbb{S}^{n+1}_1$), we obtain a second family of de Sitter structures and a map $\Omega^- : \mathcal{C}(\Sigma) \to \mathbb{S}^{n+1}_1(\Sigma \times (0, \infty))$. We say $\mathcal{M}$ is a *standard de Sitter spacetime* if $\mathcal{M} \approx \Sigma \times (0, \infty)$ and $\mathcal{M}$ is equipped with a de Sitter structure in $\Omega^+(\mathcal{C}(\Sigma)) \cup \Omega^-(\mathcal{C}(\Sigma))$. In this case, Proposition 6.1 also shows that for every $t \in (0, \infty)$, the slice $\Sigma \times \{t\}$ is a spacelike de Sitter hypersurface in the sense of §3. By construction, it is also clear that $\mathcal{M}_{max}(\Sigma \times \{t\}) = \Sigma \times (0, \infty)$. A standard de Sitter spacetime is said to be *hyperbolic* (resp. *parabolic*, *elliptic*) if it comes from a flat conformal structure of hyperbolic (resp. parabolic, elliptic) type.

CHAPTER 7

Convexity Properties

Recall that a spacelike or timelike geodesic in a Lorentz manifold may be parameterized in proportion to arclength in the usual way, while a natural choice of parameter for a null geodesic only exists up to an affine change of coordinates. The following lemma is the dual of an analogous statement for hyperbolic space; the proof itself is precisely dual to the one given by Douady in [5] for geodesics in $\mathbb{H}^2$.

Lemma 7.1 *Suppose $\alpha : [0,1] \to \mathbb{S}^n_1$ and $\beta : [0,1] \to \mathbb{S}^n_1$ are spacelike or null segments with arclength or affine parameterizations such that for all $t \in [0,1]$ we have $\tau(\alpha_t, \beta_t) > 0$. Then the function $t \mapsto -\tau(\alpha_t, \beta_t)$ is strictly convex.*

Proof: We view all points of $\mathbb{S}^n_1$ as totally geodesic hyperplanes in $\mathbb{H}^n$. With this in mind, define $\sigma_\alpha \in O(n,1)$ to be the reflection in the hyperplane $\alpha_{\frac{1}{2}}$; this isometry interchanges α_0 and α_1. Define σ_β similarly, and let δ be the common perpendicular geodesic to $\alpha_{\frac{1}{2}}$ and $\beta_{\frac{1}{2}}$. Note that δ is invariant under both σ_α and σ_β, hence also under $\sigma_\beta\sigma_\alpha$. The critical remark is that the function $\tau(-, \sigma_\beta\sigma_\alpha -)$ on de Sitter space achieves its minimum precisely on those hyperplanes perpendicular to δ (equivalently, on the $(n-2)$-plane in de Sitter space dual to δ). Thus:

$$\begin{aligned}
\tau(\alpha_1, \beta_1) + \tau(\alpha_0, \beta_0) &= \tau(\alpha_1, \beta_1) + \tau(\sigma_\alpha\alpha_1, \sigma_\beta\beta_1) \\
&= \tau(\alpha_1, \beta_1) + \tau(\sigma_\beta\sigma_\alpha\alpha_1, \beta_1) \\
&\leq \tau(\alpha_1, \sigma_\beta\sigma_\alpha\alpha_1) \\
&< \tau(\alpha_{\frac{1}{2}}, \sigma_\beta\sigma_\alpha\alpha_{\frac{1}{2}}) \qquad (7.1) \\
&= \tau(\alpha_{\frac{1}{2}}, \sigma_\beta\alpha_{\frac{1}{2}}) \\
&= \tau(\alpha_{\frac{1}{2}}, \beta_{\frac{1}{2}}) + \tau(\beta_{\frac{1}{2}}, \sigma_\beta\alpha_{\frac{1}{2}}) \\
&= 2\tau(\alpha_{\frac{1}{2}}, \beta_{\frac{1}{2}}).
\end{aligned}$$

■

A hypersurface Σ in a de Sitter spacetime is *locally convex from the future* if at every point $x \in \Sigma$ there is a null or spacelike support plane such that a neighborhood of x in Σ lies on or in the past of the support plane. Similarly, Σ is *locally strictly convex from the future* if the support planes meet Σ locally in a single point. The application of Lemma 7.1 which we will need is the following:

Proposition 7.2 *Let Σ be a spacelike de Sitter hypersurface identified with its image in $\mathcal{M}_{max}(\Sigma)$. Suppose that $H^+(\tilde{\Sigma}) \subset \overline{\mathcal{M}}_{max}(\tilde{\Sigma})$ is non-empty and locally convex from the future, with degenerate support planes corresponding to the null generators of $H^+(\tilde{\Sigma})$. Then a neighborhood in the past of $H^+(\tilde{\Sigma})$ is foliated by global Cauchy hypersurfaces for $\mathcal{M}_{max}(\tilde{\Sigma})$ which are locally strictly convex from the future and which project to global Cauchy hypersurfaces for $\mathcal{M}_{max}(\Sigma)$.*

Proof: We begin by assuming that Σ is simply-connected, and so $H^+(\Sigma)$ has the properties guaranteed by Proposition 3.3. Define the *time-to-horizon function* $\tau_{H^+} : D^+(\Sigma) \to (0, +\infty]$ by setting

$$\tau_{H^+}(x) = \sup\{\tau(x, y) \mid y \in H^+(\Sigma)\}. \tag{7.2}$$

We claim that if $\tau_{H^+}(x) = +\infty$ at any point $x \in D^+(\Sigma)$, then $\tau_{H^+} \equiv +\infty$ on all of $D^+(\Sigma)$. The set of points where τ_{H^+} equals infinity is clearly open, so consider a point x_0 such that $\tau_{H^+}(x_0) < +\infty$. The future-pointing timelike rays from x_0 all meet $H^+(\Sigma)$ in finite time, so by the local convexity of $H^+(\Sigma)$ the same holds for the future-pointing null rays from x_0. There exist local spacelike or null support planes at these intersection points, which extend slightly outside of $\overline{I^+(x_0)}$. This forces $\tau_{H^+} < +\infty$ on a neighborhood of x_0, proving the claim.

So now suppose $\tau_{H^+} \equiv +\infty$ on all of $D^+(\Sigma)$. It follows that every null generator of $H^+(\Sigma)$ is future complete; for if some null generator λ had a future endpoint $p \in H^+(\Sigma)$, then we could find a spacelike local support plane at p, which would force points in a small enough neighborhood in the past of p to satisfy $\tau_{H^+} < +\infty$. These points lie in $D^+(\Sigma)$ however, a contradiction. Given a future complete null generator λ, let N be the unique degenerate hyperplane containing $dev(\lambda)$; this is a future local support plane for the image of $H^+(\Sigma)$ by hypothesis. But if a point of $H^+(\Sigma)$ near λ develops to the past of N, we can apply the time reverse of Proposition 3.4 to see $I^-(N) = I^-(dev(\lambda))$. Thus we can find a past-pointing timelike curve in $\mathcal{M}$ joining two points of $H^+(\Sigma)$; this contradicts the achronality of $H^+(\Sigma)$. We conclude that the entire connected component of $H^+(\Sigma)$ containing λ develops into N, and so we can take as our global Cauchy hypersurfaces the level sets of a timelike horofunction for the future endpoint z of $dev(\lambda)$ on $\partial^+_\infty \mathbb{S}^n_1$. One checks easily that these surfaces are locally strictly convex from the future (they are, in fact, dual to the horospheres based at $z \in \partial_\infty \mathbb{H}^n$, which are clearly strictly convex).

Finally, we may assume that $\tau_{H^+} < +\infty$ on all of $D^+(\Sigma)$. In this case there is a continuous "farthest-point retraction" $r : D^+(\Sigma) \to H^+(\Sigma)$; the proofs of existence and continuity are dual to the analogous proofs for hyperbolic space which can be found in [6]. We claim that the level sets $\tau_{H^+}^{-1}(t)$ for small values of t foliate a neighborhood in the past of $H^+(\Sigma)$ and are locally strictly convex from the future.

Take $x \neq y$ to be two points in ${\tau_{H^+}}^{-1}[\epsilon, +\infty)$ which are spacelike-separated and close enough so that $r(x)$ and $r(y)$ lie in a locally convex neighborhood on $H^+(\Sigma)$. Let $\alpha : [0,1] \to \mathbb{S}^n_1$ be a spacelike segment joining x to y, with arclength parameter. Similarly, let β be a (possibly null) segment joining $r(x)$ to $r(y)$. It follows from Lemma 7.1 that

$$\epsilon \leq \frac{1}{2}(\tau(\alpha(0), \beta(0)) + \tau(\alpha(1), \beta(1))) < \tau(\alpha(\frac{1}{2}), \beta(\frac{1}{2})); \tag{7.3}$$

therefore if $\beta(\frac{1}{2}) \in H^+(\Sigma)$ we are done, otherwise continue the future pointing segment from $\alpha(\frac{1}{2})$ through $\beta(\frac{1}{2})$ to $H^+(\Sigma)$ to complete the proof for Σ simply-connected.

Finally, when Σ is not simply-connected, we perform the construction above for $\tilde{\Sigma}$ and note that each step is equivariant with respect to the covering transformations. ■

Proposition 7.3 *If $\Sigma \times (0, \infty)$ is a hyperbolic or parabolic standard de Sitter spacetime, then for every $t \in (0, \infty)$ the slice $\Sigma \times \{t\}$ is locally strictly convex.*

Proof: It was noted in the proof of Proposition 7.2 that the slices in a parabolic standard de Sitter spacetime are dual to horospheres in hyperbolic space and are therefore locally strictly convex. Suppose therefore that $\Sigma \times (0, \infty)$ is a hyperbolic standard de Sitter spacetime, and fix $t \in (0, \infty)$.

It was remarked above that $\mathcal{M}_{max}(\Sigma \times \{t\}) = \Sigma \times (0, \infty)$; The Cauchy horizon $H^+(\tilde{\Sigma} \times \{t\})$ in $\overline{\mathcal{M}}_{max}(\tilde{\Sigma} \times \{t\})$ is in one-to-one correspondence with the set of open round balls $U \subset \tilde{\Sigma}$ such that $U_\infty \neq \emptyset$. One sees that $H^+(\tilde{\Sigma} \times \{t\})$ satisfies the hypotheses of Proposition 7.2 in the following manner. Let U be an open round ball with $p \in U_\infty$; the failure of the local convexity property translates into the existence of a nearby open round ball which contains p, contradicting the fact that $p \in U_\infty$. The proposition is then a corollary of the previous proof, as the locally strictly convex surfaces constructed there are precisely the hypersurfaces $\tilde{\Sigma} \times \{t\}$ of constant timelike separation t from $H^+(\tilde{\Sigma} \times \{t\})$. ■

We shall see, in fact, that the proof given above of the locally convexity of of the Cauchy horizon for a standard de Sitter spacetime works quite generally; this will allow us to obtain the Classification Theorem 1.1 in §11.

CHAPTER 8

Pleated Surface Description

We will now consider the projective dual of the standard de Sitter spacetime construction. We retain the notation of §6; namely, $(D^*, \phi) = \Omega^+(D_\infty, \phi) \in \mathbb{S}_1^{n+1}(\Sigma \times (0, \infty))$ is the structure corresponding to a hyperbolic or parabolic standard de Sitter spacetime. Fix a slice $\tilde{\Sigma} \times \{t_0\}$. By Proposition 7.3, the image of $\tilde{\Sigma} \times \{t_0\}$ under D^* is locally strictly convex and spacelike, so each point has a unique spacelike support plane, which in turn defines a pole in $\mathbb{H}^{n+1}$. This gives a dual immersion of $\tilde{\Sigma} \times \{t_0\}$ onto a locally strictly convex surface in $\mathbb{H}^{n+1}$. Proceeding in this way for all $t_0 \in (0, \infty)$, we obtain an immersion $D : \tilde{\Sigma} \times (0, \infty) \to \mathbb{H}^{n+1}$ dual to D^*.

Alternatively, if the flat conformal structure is of hyperbolic type we can give a more explicit geometric construction of the map D as follows: Fix $p \in \tilde{\Sigma}$ and consider an open round ball U containing p. Thus $\mathbb{S}^n \setminus D_\infty(U)$ is a round disk which determines a closed half space in $\mathbb{H}^{n+1}$; let $\mathcal{C}_p$ be the intersection of these half spaces over all open round balls U containing p. Arguing as in the proof of Proposition 4.1, if $\mathcal{C}_p$ were empty it would follow that Σ is of elliptic or parabolic type; thus in our case $\mathcal{C}_p$ is a closed, non-empty, convex set. Following Thurston, there is a "nearest-point retraction" $r_p : \overline{\mathbb{H}^{n+1}} \to \mathcal{C}_p$ [6]. Define a map $D_0 : \tilde{\Sigma} \to \mathbb{H}^{n+1}$ by $p \mapsto r_p(D_\infty(p))$. This map is continuous and its image is locally convex; the hyperplane through $D_0(p)$ orthogonal to the geodesic ray from $D_0(p)$ to $D_\infty(p)$ is a local support plane. We can simultaneously extend both D_0 and D_∞ using the map $D : \tilde{\Sigma} \times (0, \infty) \to \overline{\mathbb{H}^{n+1}}$ which sends (p, t) to the point hyperbolic distance t along the geodesic ray from $D_0(p)$ to $D_\infty(p)$. The convexity and differentiability properties of D are of a local nature and can be obtained either by dualizing and comparing with D^*, or else by directly applying the results of [6, Ch. 1].

Proposition 8.1 *The pair (D, ϕ) defines a hyperbolic structure on $\Sigma \times (0, \infty)$. For every $t \in (0, \infty)$, the slice $\Sigma \times \{t\}$ is locally strictly convex.* ■

A *k-pleat* in $\mathbb{H}^{n+1}$ is a k-dimensional subset of $\mathbb{H}^{n+1}$ which is the convex hull of some subset of $\partial_\infty \mathbb{H}^{n+1}$. The dimension k may vary between 1 and $n + 1$. The following proposition is an immediate consequence of the definitions, and gives an alternative description of the canonical metric in terms of the path metric in $\mathbb{H}^{n+1}$. More details are available in [29].

Proposition 8.2 *Suppose Σ is a n-dimensional manifold equipped with a flat conformal structure of hyperbolic type, and let $D_0 : \tilde{\Sigma} \to \mathbb{H}^{n+1}$ be defined as above. Then D_0 maps each stratum in $\tilde{\Sigma}$ isometrically to a k-pleat in $\mathbb{H}^{n+1}$.* ■

CHAPTER 9

Bending Laminations and $\mathbb{R}$-trees

We will now consider an interesting subclass of flat conformal structures which includes the classical examples of $\mathbb{C}P^1$-structures on surfaces. Namely, we say that an n-manifold with a flat conformal structure of hyperbolic type is a *channel manifold* if all the strata in its canonical stratification are of dimension n or $n-1$. For simplicity, we will restrict our discussion to the classical case when $n = 2$ (for which the condition is automatically satisfied). The techniques of this section are due entirely to Thurston.

Suppose for the remainder of this chapter that Σ is a closed orientable surface of genus g at least two, with a flat conformal ($\mathbb{C}P^1$) structure of hyperbolic type. As usual, we write $\mathcal{S}$ for the space of strata.

Suppose that Σ has been given a Riemannian metric of constant curvature -1; a *(codimension-one) geodesic lamination* on Σ is a (possibly empty) collection $\mathfrak{L}$ of geodesics whose union is a closed subset $|\mathfrak{L}|$ of Σ. The elements of $\mathfrak{L}$ are called the *leaves* of $\mathfrak{L}$. A *measured geodesic lamination* is a pair $(\mathfrak{L}, \mu)$ consisting of a geodesic lamination $\mathfrak{L}$ and a *transverse measure* μ; that is, a function which assigns to each embedding $\alpha : [0,1] \to \Sigma$ transverse to the leaves of $\mathfrak{L}$ a finite Borel measure $\mu(\alpha)$ on the image of α satisfying the following conditions:

1. If $h : \Sigma \to \Sigma$ is isotopic to the identity via an isotopy which leaves invariant every leaf of $\mathfrak{L}$, then $\mu(\alpha) = h^*\mu(h(\alpha))$;
2. If β is a sub-path of α then $\mu(\beta)$ is the restriction of $\mu(\alpha)$;
3. The support of $\mu(\alpha)$ is precisely $\alpha([0,1]) \cap |\mathfrak{L}|$.

Following Thurston, we shall write $\int_\alpha d\mu$ for the transverse measure of a path α. The space of all measured geodesic laminations on Σ is denoted $\mathfrak{ML}(\Sigma)$.

Recall the notations of §6 and §8: From a given flat conformal structure $(D_\infty, \phi) \in \mathcal{C}(\Sigma)$ we constructed a ϕ-equivariant map $D_0 : \tilde{\Sigma} \to \mathbb{H}^{n+1}$ and a de Sitter structure $(D^*, \phi) = \Omega^+(D_\infty, \phi) \in \mathbb{S}_1^{n+1}(\Sigma \times (0,\infty))$. Kulkarni-Pinkall [29] show the existence of a canonical "reduction" of Σ; that is, a new flat conformal structure obtained by excising all embedded codimension-zero Mercator manifolds. Assume for now that this reduction has been performed. Following [6], we use the map D_0 and the path metric in $\mathbb{H}^3$ to induce a complete hyperbolic metric on Σ; let g' denote the point of Teichmüller space $\mathcal{T}(\Sigma)$ defined in this way. The set of one-dimensional pleats in the image of D_0 forms a geodesic lamination $\mathfrak{L}$ with respect the metric g'. Let $\tilde{\mathfrak{L}}$ denote the lifted geodesic lamination on $\tilde{\Sigma}$.

We associate a natural transverse measure μ to $\mathfrak{L}$ as follows. Suppose $\alpha : [0,1] \to \tilde{\Sigma}$ is an embedding transverse to the leaves of $\tilde{\mathfrak{L}}$, and suppose $s, t \in [0,1]$ are such that $\alpha(s)$ and $\alpha(t)$ do not lie in $|\tilde{\mathfrak{L}}|$. Define $\angle_\alpha(s,t)$ to be the dihedral angle between the (unique) support planes at $D_0(\alpha(s))$ and $D_0(\alpha(t))$ and set

$$\int_\alpha d\mu = \inf\{\angle_\alpha(t_0,t_1) + \cdots + \angle_\alpha(t_{n-1},t_n)\}, \tag{9.1}$$

where the infimum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that no $\alpha(t_j)$ lies in $|\tilde{\mathfrak{L}}|$. When the flat conformal structure is not assumed to be reduced, the associated geodesic lamination may contain isolated leaves. The transverse measure across such a leaf is defined to be the "width" of the corresponding Mercator manifold in Σ. We call the resulting measured geodesic lamination $(\mathfrak{L}, \mu)$ the *bending lamination* associated to the channel manifold [1], [6]. Thurston (unpublished) has shown that the map

$$\begin{aligned} \Theta :\ \mathcal{C}(\Sigma) &\to \mathcal{T}(\Sigma) \times \mathfrak{ML}(\Sigma) \\ (D_\infty, \phi) &\mapsto (g', (\mathfrak{L}, \mu)) \end{aligned} \tag{9.2}$$

is a bijection (in fact a homeomorphism with respect to appropriately defined topologies); his proof is reconstructed in [23], while Labourie has obtained a completely different proof in [31]. Kulkarni-Pinkall [29] have obtained similar statements in higher dimensions.

We will need the following useful description of the closed leaves in a bending lamination:

Lemma 9.1 *Given a measured geodesic lamination $(\mathfrak{L}, \mu)$ on Σ and a leaf $\lambda \in \mathfrak{L}$, the following are equivalent:*

1. *λ is isolated*
2. *λ has positive transverse measure*
3. *λ is closed*

Furthermore, if $(\mathfrak{L}, \mu)$ is a bending lamination, then the holonomy of a closed leaf is infinite cyclic and purely hyperbolic.

Proof: It is clear that (1) implies (2) because the support of μ is all of $|\mathfrak{L}|$, and (2) implies (3) or else there would be an embedded path with infinite transverse measure near an accumulation point of λ. The argument of Lemma 4.6 of [3] shows that if λ is closed, then it has a regular neighborhood N such that any leaf which intersects $N \setminus \lambda$ is isolated and asymptotic to λ. We have shown on the other hand that isolated leaves are closed, and so λ is the only leaf intersecting N.

If λ is a closed leaf in a bending lamination and $\tilde{\lambda}$ is a lift to the universal cover, then $\tilde{\lambda}$ maps homeomorphically onto a one-dimensional pleat in $\mathbb{H}^3$. Hence the holonomy of λ is infinite cyclic, generated by a loxodromic element leaving the pleat invariant. The complementary regions on either side are also invariant, so the holonomy generator has no rotational component. ■

An *arc* in a topological space is a subspace homeomorphic to a compact interval of $\mathbb{R}$. An $\mathbb{R}$-*tree* is a metric space Y such that for any $x \neq y \in Y$, there is a unique arc $A \subseteq Y$ with endpoints x and y, and A is isometric to a compact interval of $\mathbb{R}$.

Given a measured geodesic lamination $(\mathfrak{L}, \mu)$ on Σ, and a lift $(\tilde{\mathfrak{L}}, \tilde{\mu})$ on $\tilde{\Sigma}$, we let C be the set of components of $\tilde{\Sigma} \setminus |\tilde{\mathfrak{L}}|$. For c_0 and c_1 in C, choose points $y_0 \in c_0$ and $y_1 \in c_1$, and consider the geodesic segment $[y_0, y_1]$ between them. This segment is transverse to the leaves of $\tilde{\mathfrak{L}}$, and clearly cannot cross a given leaf more than once. Using the fact that $|\tilde{\mathfrak{L}}|$ is nowhere dense in $\tilde{\Sigma}$ [3], [45, §8.5], an easy argument verifies that the transverse measure $\int_{[y_0,y_1]} d\mu$ is independent of the choice of y_0 and y_1. Setting $d(c_0, c_1) = \int_{[y_0,y_1]} d\mu$, it is again easily verified that d defines a $\pi_1(\Sigma)$-invariant metric on C.

Lemma 9.2 *[40] There exist an $\mathbb{R}$-tree T and an isometric embedding $\psi : C \hookrightarrow T$ such that*

1. *The smallest subtree containing $\psi(C)$ is T itself*
2. *Any $t \in T \setminus \psi(C)$ separates T into exactly two components*
3. *The action of $\pi_1(\Sigma)$ on C extends (uniquely) to an isometric action on T*

Furthermore, if (T, ψ) and (T', ψ') both satisfy the conditions above, then there exists a $\pi_1(\Sigma)$-equivariant isometry $\iota : T \to T'$ such that $\iota \circ \psi = \psi'$. ■

The $\mathbb{R}$-tree given by Lemma 9.2 is called the *dual tree* to the measured geodesic lamination $(\mathfrak{L}, \mu)$. A general development of the theory of $\mathbb{R}$-trees and codimension-one measured laminations is contained in a fundamental series of papers by Morgan and Shalen [37, 38, 39].

The next proposition ties together the dual descriptions of the canonical stratification by showing that the space of strata $\mathcal{S}$ is an $\mathbb{R}$-tree, isometric to the abstract dual tree T arising from the bending lamination in $\mathbb{H}^3$.

Proposition 9.3 *Suppose Σ is a closed orientable surface of genus at least two, with a flat conformal structure of hyperbolic type, space of strata $\mathcal{S}$, and associated bending lamination $(\mathfrak{L}, \mu)$. Then $\mathcal{S}$ and the dual tree to $(\mathfrak{L}, \mu)$ are naturally isometric.*

Proof: Let the notation be chosen as in the statement of the proposition, and let C denote the space of complementary regions of $\tilde{\mathfrak{L}}$, with metric d_C as defined above.

For each $p \in \tilde{\Sigma}$, there is timelike geodesic joining $D_0^*(C(U_p))$ and $D_\infty(p)$ (as in the definition of D^* in §6). Together, these geodesics define a deformation retraction of $\tilde{\Sigma}$ onto $\mathcal{S}$. Hence $\mathcal{S}$ is a connected, simply-connected, one-dimensional metric space, in which the distance between two points is given by the path of minimal length joining them. It follows that $\mathcal{S}$ is an $\mathbb{R}$-tree.

There is an obvious $\pi_1(\Sigma)$-equivariant inclusion of C into $\mathcal{S}$ which we will write as ψ; by the uniqueness statement of Lemma 9.2, it will suffice to show that ψ satisfies the various conditions of that lemma. The complement of $\psi(C)$ in $\mathcal{S}$ is precisely the collection of one-dimensional strata; since any such stratum divides $\tilde{\Sigma}$ into two components both of which contain two-dimensional strata, conditions (1) and (2) hold. It remains to verify that ψ is an isometric embedding.

For simplicity, we will assume we are in the (generic) case of a *reduced* flat conformal structure; the necessary modifications for handling non-trivial Mercator manifolds are left to the reader. Let $\beta : [0,1] \to \tilde{\Sigma}$ be a geodesic path transverse to the leaves of $\tilde{\mathfrak{L}}$ joining the complementary regions c_0 and c_1, and write c_s for the complementary region containing $\beta(s)$ whenever $\beta(s) \notin |\tilde{\mathfrak{L}}|$. Since $\mathcal{S}$ is an $\mathbb{R}$-tree, there is a unique arc $[\psi(c_0), \psi(c_1)]$ joining $\psi(c_0)$ and $\psi(c_1)$ whose length is obtained by means of the map $D_0^* : \mathcal{S} \to (\mathbb{H}^3)^*$. The bending angle $\angle_\beta(s,t)$ between two complementary regions is (by definition) computed by applying D_0 and measuring the dihedral angle between the corresponding pleats; equivalently we may apply the dual map D_0^* and considering the spacelike distance in $\mathbb{S}_1^3$. Therefore we have:

$$
\begin{aligned}
d_C(c_0, c_1) &= \int_\beta d\mu \\
&= \inf\{\angle_\beta(t_0, t_1) + \cdots + \angle_\beta(t_{k-1}, t_k)\} \\
&= \inf\{L[D_0^*(\psi(c_{t_0})), D_0^*(\psi(c_{t_1}))] + \cdots + L[D_0^*(\psi(c_{t_{k-1}})), D_0^*(\psi(c_{t_k}))]\} \\
&= L(D_0^*([\psi(c_0), \psi(c_1)])) \\
&= d_{\mathcal{S}}(\psi(c_0), \psi(c_1)).
\end{aligned}
\tag{9.3}
$$

■

CHAPTER 10

Horizons in Standard de Sitter Spacetimes

We begin the discussion of Theorem 1.2 by constructing an interesting counterexample in the two-dimensional case. It follows from Euler characteristic considerations that any two-dimensional spacetime-bordism is homeomorphic to an annulus; nevertheless we have found a family of de Sitter annuli which contain non-trivial Cauchy horizons.

Suppose $\tau \in SO_0(2,1)$ is a hyperbolic element, and fix a ruling of $\mathbb{S}^2_1$ by null lines. This defines a pair of disjoint null lines in $\mathbb{S}^2_1$ joining the two fixed points of τ in $\partial^+_\infty \mathbb{S}^2_1$ with the ones in $\partial^-_\infty \mathbb{S}^2_1$. The element τ acts freely and properly discontinuously on an open region $\mathcal{U}$ bounded by these two lines (compare figure 10.1, where for simplicity we have indicated this situation in the universal cover of $\mathbb{S}^2_1$). The quotient $\mathcal{U}/\langle\tau\rangle$ is a de Sitter annulus with two non-trivial Cauchy horizons corresponding to the two null rays in $\mathcal{U}$ left invariant by τ. One can construct similar examples which contain no closed timelike curves by choosing τ to be parabolic. See figure 10.2.

In dimension three, similar examples arise when there are open subsets in the Cauchy horizon which are foliated by null generators. As the next result shows, for a standard de Sitter spacetime this occurs precisely when there is a closed leaf in the associated measured geodesic lamination, which gives rise to an embedded codimension-zero Hopf manifold.

Theorem 1.3 *Suppose Σ is a closed, orientable surface with a $\mathbb{C}P^1$-structure and let $\mathcal{M} \approx \Sigma \times (0,\infty)$ be the associated standard de Sitter spacetime. Then $\mathcal{M}$ is a*

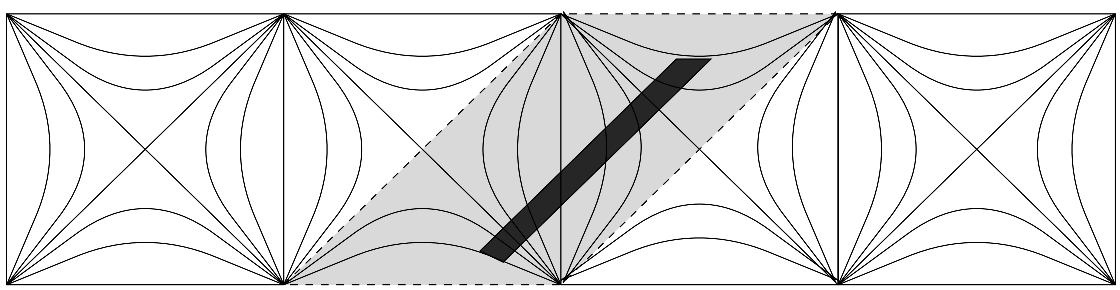

Figure 10.1: The lightly shaded region indicates the universal cover of an open 1+1 de Sitter annulus with hyperbolic holonomy which is not a domain of dependence. The darker region is a fundamental domain for compact annulus with spacelike boundary.

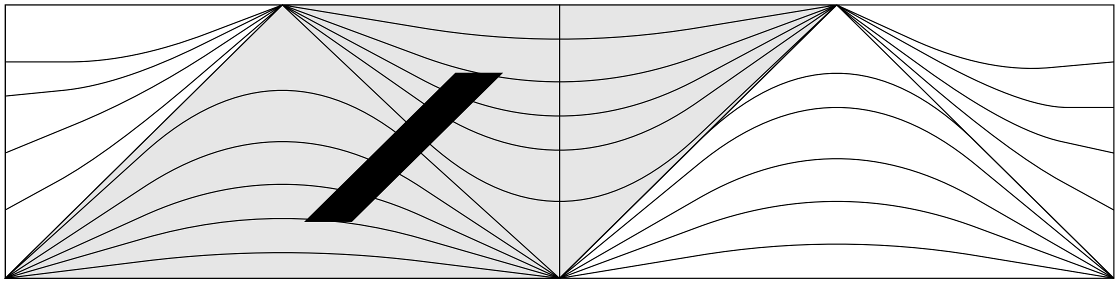

Figure 10.2: The lightly shaded region indicates the universal cover of an open 1+1 de Sitter annulus with parabolic holonomy which is not a domain of dependence. The darker region is a fundamental domain for compact annulus with spacelike boundary and no closed timelike curves.

domain of dependence, and embeds in a strictly larger de Sitter spacetime if and only if Σ *contains a codimension-zero Hopf manifold.*

Proof: Suppose $\mathcal{M} \approx \Sigma \times (0, \infty)$ is a standard de Sitter spacetime, with de Sitter structure $(D^*, \phi) = \Omega^+(D_\infty, \phi) \in \mathbb{S}^3_1(\mathcal{M})$ as constructed in §6. We have shown in Proposition 6.1 that $\mathcal{M}$ is a domain of dependence, so we need only examine when $\mathcal{M}$ can be extended to a strictly larger de Sitter spacetime.

The theorem is clear for elliptic standard de Sitter spacetimes. If $\mathcal{M}$ is a parabolic standard de Sitter spacetime, then the image of ϕ consists entirely of parabolic elements of $SO_0(3,1)$ fixing a common point $z \in \partial^+_\infty \mathbb{S}^3_1$. These elements leave invariant each null line in $I^+(z)$; it follows that no two distinct generators of $\phi(\pi_1(\Sigma))$ can act discontinuously on $I^+(z)$. Hence $\mathcal{M}$ cannot be extended to a strictly larger de Sitter spacetime (our parabolic $(1+1)$-dimensional example indicates the necessity of having at least two generators of the holonomy image). For the remainder of the proof we shall assume $\mathcal{M}$ is a hyperbolic standard de Sitter spacetime, with space of strata $\mathcal{S}$ for $\tilde{\Sigma}$.

Suppose first that $\mathcal{M} \subsetneq \mathcal{M}'$ for some de Sitter spacetime $\mathcal{M}'$; let $D' : \tilde{\mathcal{M}}' \to \mathbb{S}^3_1$ be the developing map extending D^*. Since the flat conformal structure is of hyperbolic type, the genus g of Σ cannot be zero, and if $g = 1$, then Σ is a Hopf manifold and we are done. Thus we may assume g is at least two, and so there is an associated bending lamination $(\mathfrak{L}, \mu) \in \mathfrak{ML}(\Sigma)$ as in §9. For any $t \in (0, \infty)$ we have $H^+(\Sigma \times \{t\}) \neq \emptyset$ in $\mathcal{M}'$ since $\mathcal{M}$ is maximal as a domain of dependence. Each point of $H^+(\tilde{\Sigma} \times \{t\})$ can be identified with an open round ball U for the flat conformal structure on Σ satisfying $U_\infty \neq \emptyset$. Recall that $\mathcal{S}$ embeds in the space of open round balls for Σ; denote by $\mathcal{S}_H$ the intersection of $\mathcal{S}$ and $H^+(\tilde{\Sigma} \times \{t\})$ and by $\mathcal{S}_T$ the collection of two-dimensional strata in $\mathcal{S}_H$.

Choose an open ball $\mathcal{W} \subset \tilde{\mathcal{M}}'$ around a point of $H^+(\tilde{\Sigma} \times \{t\})$, small enough so that the projection to $\mathcal{M}'$ and the developing map D' are both one-to-one when restricted to $\mathcal{W}$.

Claim 1: $\mathcal{W} \cap \mathcal{S}_T = \emptyset$.

First suppose that there are infinitely many two-dimensional strata c_j lying in $\mathcal{W}$. All two-dimensional strata are complete hyperbolic surfaces and therefore each has area at least π, so there can be at most $4g - 4$ such strata on Σ by Gauss-Bonnet. Thus there exists a subsequence $\{c_{j_k}\}$ and corresponding elements $\{\gamma_{j_k}\}$ in $\pi_1(\Sigma)$ with $c_{j_k} = \gamma_{j_k} \cdot c$ for some fixed $c \in \mathcal{S}_T$. This contradicts our choice of $\mathcal{W}$. Now suppose there is an isolated point $c_0 \in \mathcal{W} \cap \mathcal{S}_T$. One possibility is that there are no one-dimensional strata, in which case c_0 is fixed by every element of $\phi(\pi_1(\Sigma))$, contradicting proper discontinuity of the action. On the other hand, if $\mathfrak{L} \neq \emptyset$, the boundary leaves of c_0 are isolated, hence are simple closed geodesics with non-trivial holonomy by Lemma 9.1. But these holonomy elements all fix c_0, again contradicting proper discontinuity. This proves claim 1.

Claim 2: $\mathcal{W} \cap \mathcal{S}_H = \emptyset$.

Suppose not; so there is some $t \in \mathcal{W} \cap \mathcal{S}_H$. Using claim 1, there is a neighborhood of t in $\mathcal{S}$ consisting only of one-dimensional strata. This corresponds to a non-trivial embedded Hopf manifold, and in turn to an isolated simple closed geodesic in $\mathfrak{L}$ with positive transverse measure. The holonomy of this geodesic is non-trivial and purely hyperbolic, from which we see that it fixes a family one-dimensional strata near t. This contradicts proper discontinuity, thereby proving claim 2.

Now suppose $x \in H^+(\tilde{\Sigma} \times \{t\})$. It was shown earlier that the Cauchy horizon for a standard de Sitter spacetime is locally convex, so there are local support planes at x. The degenerate support planes at x each define a point of $\partial^-_\infty \mathbb{S}^3_1$; let C be the convex hull of these points in $\mathbb{H}^3$. The set C is dual to the set of spacelike support planes at x. When x has at least three degenerate support planes, C is a complementary region of the bending lamination (and so $x \in \mathcal{S}_T$), while if x has exactly two degenerate support planes, C is a leaf of the bending lamination (and $x \in \mathcal{S}_H$). Thus each point in the remainder of the frontier $F \setminus \delta(T)$ has a unique degenerate support plane, and by the above results, there is an induced foliation of $\mathcal{W}$ by null lines.

Case 1: $\mathfrak{L} = \emptyset$.

The holonomy representation ϕ is Fuchsian, preserving a plane $\Delta \subset \mathbb{H}^3$, and the frontier F of the image of D is a past-pointing null cone from the pole $\Delta^* \in (\mathbb{H}^3)^*$. Let $\Gamma = \phi(\pi_1(\Sigma))$. There is a natural identification between $F \setminus \{\Delta^*\}$ and the set of (left) horocycles in Δ. Hedlund's theorem [21] shows that for any (left) horocycle H viewed as a subset of the unit tangent bundle $UT(\Delta)$, the set ΓH is dense in $UT(\Delta)$. It follows immediately from the above identification that the action of Γ on F is not properly discontinuous at any point.

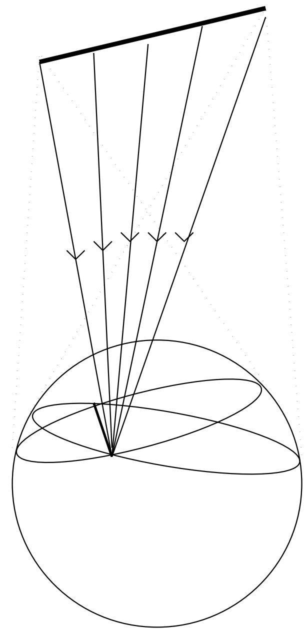

Figure 10.3: “Bending” along an isolated geodesic in hyperbolic space is equivalent to “grafting” a θ-annulus on $\mathbb{C}P^1$, and also to inserting a region foliated by asymptotic null rays in de Sitter space (“stretching”).

Case 2: $\mathfrak{L} \neq \emptyset$.

The proof in this case is analogous to Proposition 14 of [35]; one shows that the set of basepoints of null rays foliating $\mathcal{W}$ is a spacelike segment dual to an isolated leaf. The isolated leaf has positive transverse measure by Lemma 9.1, and therefore corresponds to a non-trivial embedded Hopf manifold. This completes the first half of Theorem 1.3.

Conversely, suppose there is an embedded codimension-zero Hopf manifold in Σ. The invariant axis of the holonomy of this submanifold projects to a closed leaf λ in $\mathfrak{L}$. Let $\tilde{\lambda}$ be a lift of λ to the universal cover $\tilde{\Sigma}$, and let c_0 and c_1 be the neighboring complementary regions. Then as above, there is a unique segment $[\psi(c_0), \psi(c_1)] \subseteq \mathcal{S}$, which maps under D_0^* to a spacelike line segment of length equal to the transverse measure across λ. The holonomy of λ, being purely hyperbolic, fixes this segment pointwise. From each point of $D_0^*([\psi(c_0), \psi(c_1)])$, there are two null rays meeting the fixed points at infinity of $\phi(\langle\lambda\rangle)$, see figure 10.3. This is completely analogous to the two-dimensional example constructed at the beginning of this chapter. We may extend $\mathcal{M}$ to $\mathcal{M}'$ so that $\tilde{\mathcal{M}}' \setminus \tilde{\mathcal{M}}$ develops into a small holonomy-invariant neighborhood of the translates of one of these rays. Thus we have embedded $\mathcal{M}$ in a strictly larger de Sitter spacetime (which is necessarily non-compact and contains closed causal curves). This completes the

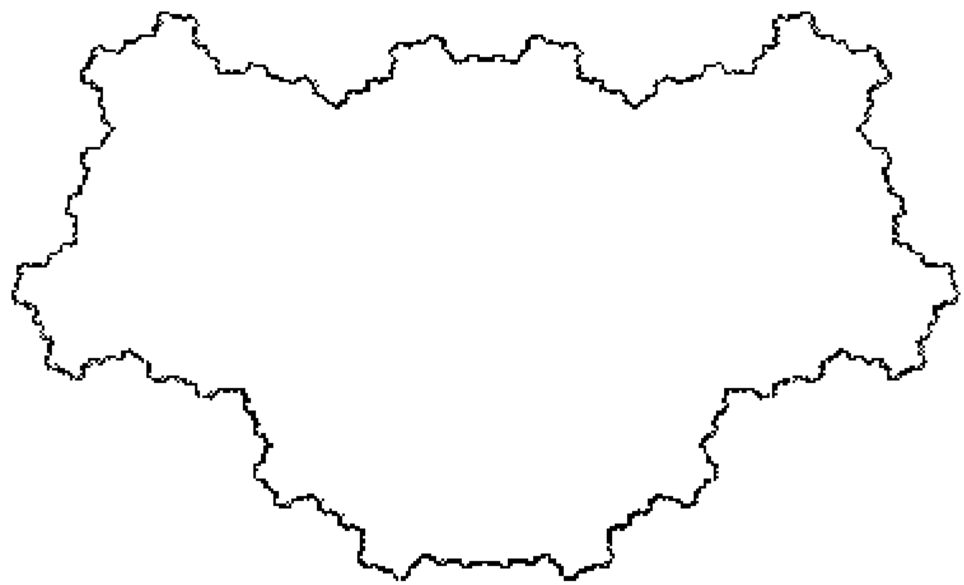

Figure 10.4: The effect on the limit set of bending a (fictional) Fuchsian group on the lifts of a single simple closed geodesic is illustrated here; these are the so-called "Mickey Mouse" examples of Thurston.

proof of Theorem 1.3. ■

CHAPTER 11

Proofs of Main Theorems

The first proposition is the key observation from which our main results are derived.

Proposition 11.1 *Suppose $\mathcal{M}$ is a spacetime of constant curvature, and $\Sigma \subset \mathcal{M}$ is a closed achronal spacelike hypersurface. Suppose further that $H^+(\Sigma)$ is non-empty, and the null generators of $H^+(\Sigma)$ are past complete. Then $H^+(\Sigma)$ is locally convex from the future, with degenerate support planes corresponding to the null generators of $H^+(\Sigma)$.*

Proof: Write $dev : \tilde{\mathcal{M}} \to X$ for the developing map into the appropriate constant curvature model space $X = \mathbb{S}^n_1, \mathbb{R}^n_1$, or $\mathbb{H}^n_1$. Take $x \in H^+(\Sigma)$; by Lemma 3.2 there exists an inextendible, past-pointing null generator β starting at x and lying entirely within $H^+(\Sigma)$. Choose a lift $\tilde{x} \in \tilde{\mathcal{M}}$ of x, and let $\tilde{\beta}$ be the lift of β starting at $\tilde{x}$. There is a unique degenerate hyperplane $N \subset X$ containing $dev(\tilde{\beta})$; we claim N is a local support plane for the developing image of a neighborhood of $\tilde{x}$ in $H^+(\tilde{\Sigma}) = \widetilde{H^+(\Sigma)}$. Suppose not, so there is a point $p \in I^+(N)$ in the developing image of a small neighborhood in $H^+(\tilde{\Sigma})$ of $\tilde{x}$. But Proposition 3.4 implies that $p \in I^+(dev(\tilde{\beta}))$; choosing p close enough to $dev(\tilde{x})$, we assure the existence of a past-pointing timelike curve from p to a point of $dev(\tilde{\beta})$ which lies entirely within the developing image of a small neighborhood of $\tilde{\beta}$. This contradicts the fact that $H^+(\Sigma)$ is achronal, proving the proposition. ■

We have modified the statement of Theorem 1.1, taking advantage of the formalism of §3:

Theorem 1.1 *If Σ is a spacelike de Sitter hypersurface, then $\mathcal{M}_{max}(\Sigma)$ is a standard de Sitter spacetime.*

Proof: If $\mathcal{M}_{max}(\Sigma)$ is both past and future complete, it follows that it is isometric to a manifold of the form $\mathbb{S}^n_1/\Gamma$ for some finite subgroup of Γ of $SO_0(n,1)$ (see the Appendix). It is well-known [49, 11.2] that all such subgroups are conjugate into the maximal compact subgroup $SO(n)$ of $SO_0(n,1)$, and therefore Γ acts freely and isometrically on $\partial^+_\infty \mathbb{S}^{n+1}_1$ and $\partial^-_\infty \mathbb{S}^{n+1}_1$. Thus $\mathcal{M}_{max}(\Sigma)$ is a standard de Sitter spacetime arising from the spherical space form so-defined.

Assume now, without loss of generality, that $\mathcal{M}_{max}(\Sigma)$ fails to be future complete, and apply Proposition 3.3 to embed $\mathcal{M}_{max}(\tilde{\Sigma})$ in $\overline{\mathcal{M}}_{max}(\tilde{\Sigma})$ so that $H^+(\tilde{\Sigma}) \neq \emptyset$ and all null generators are past complete. Combining Propositions 11.1 and 7.2

shows that there is a global Cauchy hypersurface Σ' for $D^+(\Sigma) = \mathcal{M}_{max}(\Sigma)$ which is locally strictly convex from the future. Thus $\mathcal{M}_{max}(\Sigma) \subseteq \mathcal{M}_{max}(\Sigma')$. Under this inclusion Σ becomes a global Cauchy hypersurface for the spacetime $\mathcal{M}_{max}(\Sigma')$ since any causal curve meets Σ' and therefore Σ also; hence $\mathcal{M}_{max}(\Sigma) = \mathcal{M}_{max}(\Sigma')$.

Because Σ' is locally strictly convex and spacelike, we can define a corresponding flat conformal structure on Σ' by the Gauss map; i.e. following the unique timelike normal line at each point of the developing image of $\tilde{\Sigma}'$ to past infinity defines an equivariant developing map $D_\infty : \tilde{\Sigma}' \to \partial^-_\infty \mathbb{S}^{n+1}_1$. The standard de Sitter spacetime corresponding to this flat conformal structure contains Σ' as a global Cauchy hypersurface, and so it equals $\mathcal{M}_{max}(\Sigma')$. Thus $\mathcal{M}_{max}(\Sigma)$ is a standard de Sitter spacetime. ■

Theorem 1.2 *Every three-dimensional de Sitter spacetime-bordism is a domain of dependence, with the exception of those standard de Sitter spacetimes arising from closed two-dimensional Hopf manifolds.*

Proof: We begin by assuming that $\mathcal{M}$ is a de Sitter spacetime-bordism which is not a domain of dependence. The boundary of $\mathcal{M}$ is non-empty; without loss of generality assume $\partial^-\mathcal{M} \neq \emptyset$ and so there is a component $\Sigma \subseteq \partial^-\mathcal{M}$ such that $H^+(\Sigma) \neq \emptyset$. As noted in §3, there is an isometric embedding $\iota : D^+(\Sigma) \hookrightarrow \mathcal{M}_{max}(\Sigma)$. By the Classification Theorem 1.1, $\mathcal{M}_{max}(\Sigma)$ is a standard de Sitter spacetime, arising from some flat conformal structure on Σ. The embedding ι extends to an isometric embedding $\iota' : \overline{D^+(\tilde{\Sigma})} \hookrightarrow \overline{\mathcal{M}}_{max}(\tilde{\Sigma})$. The action of $\pi_1(\Sigma)$ on $\overline{\mathcal{M}}_{max}(\tilde{\Sigma})$ is described by Theorem 1.3, which shows in particular that the flat conformal structure on Σ is of hyperbolic type and contains a codimension-zero Hopf manifold Σ_0. Let H_0 be the region in the future Cauchy horizon foliated by null rays associated to Σ_0. The proof of Theorem 1.3 shows that any holonomy elements which are not in the infinite cyclic subgroup corresponding to Σ_0 cannot act discontinuously on $\overline{H}_0$. We conclude that Σ is itself a Hopf manifold, completing the proof. ■

APPENDIX A

Survey of Classification Theorems

In this appendix, we will provide statements of many of the theorems concerning the classification of constant curvature Lorentz metrics on compact manifolds, concentrating particularly on dimension three. We first consider the case of closed (compact, no boundary) spacetimes. With no curvature assumptions, such a spacetime need not be complete; the classical Clifton-Pohl torus serves as the standard counterexample (see [41, Ch. 7]). The key rigidity result for constant curvature spacetimes is as follows:

Theorem A.1 *A closed spacetime of constant curvature is (geodesically) complete.*

This was first proved by Carrière [2] in the flat case and extended to the anti-de Sitter case by Mess in [35]. A unified proof containing all three cases has recently been given by Klingler [26].

It follows from Theorem A.1 that any closed spacetime of constant curvature is of the form X/Γ, where X is the appropriate constant curvature model space and Γ is a discrete cocompact group of isometries of X. With this in mind, de Sitter manifolds become particularly easy to classify:

Proposition A.2 *A complete de Sitter spacetime has finite fundamental group.*

Proof: [49] Let $\mathbb{S}^n_1/\Gamma$ be a complete de Sitter spacetime, and suppose Γ is infinite. We have an embedding of flat Euclidean space $\mathbb{E}^n \subset \mathbb{R}^{n+1}_1$ as the set of vectors such that $x_{n+1} = 0$, and so $\mathbb{S}^{n-1} = \mathbb{S}^n_1 \cap \mathbb{E}^n$. For any transformation $A \in GL(n+1, \mathbb{R})$ we have that $\dim(\mathbb{E}^n \cap A(\mathbb{E}^n)) \geq n-1$, and thus for every $\gamma \in \Gamma$, we have $\gamma(\mathbb{S}^{n-1}) \cap \mathbb{S}^{n-1} \neq \emptyset$. By compactness of $\mathbb{S}^{n-1}$, there exist infinitely many distinct elements $\gamma_i \in \Gamma$ and points $x_i \in \mathbb{S}^{n-1}$ such that the sequence $\{\gamma_i x_i\}$ converges to a point $y \in \mathbb{S}^{n-1}$. Subsequence so that $\{x_i\}$ converges to a point $x \in \mathbb{S}^{n-1}$. Thus $\{\gamma_i x\}$ converges to y, contradicting proper discontinuity at x. ■

Corollary A.3 *There are no closed de Sitter spacetimes.* ■

Next, we consider the flat case. The group of isometries of $\mathbb{R}^n_1$ is represented in the usual way as a semi-direct product of $O(n-1,1)$ and $\mathbb{R}^n_1$; we write $L : Isom(\mathbb{R}^n_1) \to O(n-1,1)$ for the homomorphism projecting to the linear part. Given a subgroup Γ of $Isom(\mathbb{R}^n_1)$, define $T(\Gamma) = \ker L\big|_{\Gamma}$; we call $T(\Gamma)$ the *translational* *subgroup* of Γ. The image $L(\Gamma)$ is the *linear holonomy* of Γ. We have the following short exact sequence:

$$1 \to T(\Gamma) \to \Gamma \to L(\Gamma) \to 1 \tag{A.1}$$

Proposition A.4 *If $\mathcal{M}$ is a closed, three-dimensional, flat spacetime then $\pi_1\mathcal{M}$ is virtually solvable.*

Proof: A closed, three-dimensional, flat spacetime is complete by Theorem A.1 and will therefore be identified with the quotient $\mathbb{R}^3_1/\Gamma$ for a discrete subgroup $\Gamma \subset Isom(\mathbb{R}^3_1)$. Arguing as in [9] or [35], if the linear holonomy $L(\Gamma)$ fails to be discrete, then Γ is automatically solvable. On the other hand, if $L(\Gamma)$ is discrete, we can break the proof up into cases depending on the rank of the translational subgroup $T(\Gamma)$. If $rank(T(\Gamma)) = 3$, it follows that $\mathcal{M}$ is finitely covered by a torus, and so clearly Γ is virtually solvable. Similarly, if $rank(T(\Gamma)) = 1$ or 2, then $L(\Gamma)$ leaves invariant a non-trivial linear subspace in $\mathbb{R}^3_1$, and again Γ is solvable. In the last case, $\Gamma \cong L(\Gamma)$, which would imply the cohomological dimension of Γ is 2, a contradiction. ■

This is an extremely easy special case of the main theorem of [17], which asserts that the fundamental group of a closed flat spacetime of any dimension is virtually polycyclic. In dimension three it follows from Proposition A.4 that a closed flat spacetime is finitely covered by a torus bundle over $\mathbb{S}^1$ (see [7]), and hence that it is modeled on a three-dimensional solvable Lie group (Nil, $Solv$, or $\mathbb{E}^3$). The classification of the possible lattices Γ up to affine isomorphism is given in [9].

For the anti-de Sitter case, we note that there is an identification of $\mathbb{H}^3_1$ with $SL(2,\mathbb{R})$, where the indefinite metric is given by an appropriate multiple of the Killing form. The identity component of the isometry group of $\mathbb{H}^3_1$ is $SO_0(2,2)$, and with respect to the above identification, we have an isomorphism

$$SO_0(2,2) \cong (SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/(-I,-I) \tag{A.2}$$

where the action is by left and right multiplication on $SL(2,\mathbb{R})$. The results of [30] and [14] have the following consequence:

Theorem A.5 *A closed, three-dimensional, anti-de Sitter spacetime is homeomorphic to either a Seifert fibered space with non-zero Euler number, or a connected sum of lens spaces. Conversely, any such three-manifold admits an anti-de Sitter structure.* ■

Finally we outline very briefly the results obtained by Mess in the case of flat and anti-de Sitter spacetime-bordisms. The first result is obtained by modifying Carrière's proof of Theorem A.1.

Theorem A.6 *Every three-dimensional flat or anti-de Sitter spacetime-bordism is homeomorphic to a product $\Sigma \times [0,1]$ with spacelike slices $\Sigma \times \{t\}$.* ■

The classification theory in the flat case is similar in many respects to our development in the de Sitter case. When the genus of Σ is at least two, the standard flat spacetimes are by definition those domains of dependence which contain a strictly convex global Cauchy hypersurface. As in the de Sitter case, these standard examples have convex Cauchy horizons. This enables one to parameterize the standard flat spacetimes by the space of all measured geodesic laminations $\mathcal{T}(\Sigma)\times\mathfrak{ML}(\Sigma)$. Every three-dimensional flat spacetime-bordism which is a domain of dependence isometrically embeds in a standard flat spacetime.

The standard anti-de Sitter spacetimes homeomorphic to $\Sigma \times [0,1]$ are parameterized by $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$, where for any pair of Fuchsian representations (ρ_L, ρ_R) Mess constructs a corresponding anti-de Sitter spacetime with this holonomy in $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. Once again, the classification theorem states that any anti-de Sitter spacetime which is a small regular neighborhood of a compact spacelike surface isometrically embeds in a standard anti-de Sitter spacetime. The reader is encouraged to consult [35] for details.

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